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Functional Calculus for Infinitesimal Generators of Holomorphic Semigroups

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We give a functional calculus formula for infinitesimal generators of holomorphic semigroups of operators on Banach spaces, which involves the Bochner–Riesz kernels of such generators. The rate of smoothness of operating functions is related to the exponent of the growth on vertical lines of the operator norm of the semigroup. The strength of the formula is tested on Poisson and Gauss semigroups in $L^1(\mathbf{R}^n)$ and $L^1(G)$, for a stratified Lie group G. We give also a self-contained theory of smooth absolutely continuous functions on the half line $[0, \infty)$.

Key Words: holomorphic semigroups; functional calculus; multipliers; stratified Lie groups; fractional derivatives; Leibniz formula.

1. INTRODUCTION

Two fundamental and well-known functional calculi that may be developed for a closed, densely defined operator A on a Banach space X are the *holomorphic calculus*, which works for arbitrary A, and the L^∞-*calculus* associated to operators with a spectral decomposition—e.g., self adjoint operators on Hilbert spaces. Very often, these calculi are used as starting

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points to build other intermediate types of calculi for A, in accordance with specific properties of the operator.

It is also possible to define a functional calculus on the basis of the Fourier inversion formula. Suppose for a moment that A is such that e^{isA} is somehow defined as a bounded operator on X for every $s \in \mathbf{R}$. Then the correspondence $f \to \int_{\mathbf{R}} \hat{f}(s) e^{isA}\, ds$ provides a functional calculus for A under the assumption that $\hat{f}(s)\, \|e^{isA}\|$ is integrable on $\mathbf{R}$ (as usual, $\hat{f}$ is the Fourier transform of f and $\|e^{isA}\|$ is the norm of the operator e^{isA}). In many cases the growth of $\|e^{isA}\|$ at infinity turns out to be polynomial, of degree d, say, so that f is usually taken to be $d+2$ times differentiable. In this form, the calculus has been revealed to be fruitful (e.g., [B, HJ, K, Ka]), although it is to be noticed that, in this direction, generally neither is the calculus expressed in terms of derivatives themselves nor is the relation between the "size" of the polynomial growth of $\|e^{isA}\|$ and the necessary degree of differentiability of the operating functions cleared up. Furthermore, the method so considered cannot be directly applied when the operators e^{isA} are unbounded, which happens for important cases like those of differential operators, for instance (it is well known that $e^{is\Delta}$, where Δ is the Laplacian on $\mathbf{R}^n$, is not bounded on $L^p(\mathbf{R}^n)$ unless $p=2$). Nevertheless, there is a wide range of examples where one can think of e^{isA} as "boundary values" of analytic semigroups $a^z := e^{-zA}$ of bounded operators on $\Re z > 0$, with polynomial growth on vertical lines in $\Re z > 0$. The Gauss and Poisson semigroups on $\mathbf{R}^n$ are prototypes of this kind of semigroup.

Thus it appears that the calculus for differentiable functions based on the (formal) inverse Fourier transform would deserve further investigation, in order to extend the method, so we can apply it to infinitesimal generators of semigroups as above, and searching, if possible, for a formula containing explicit derivatives, accurately related to the growth on vertical lines of the semigroup. This is one of the goals of this paper.

To deal with the above kinds of questions, fractional derivation seemed to be an appropriate tool, after a long history connected with functional calculus and semigroups [HP, SKM]. In recent times, the reproducing formula

$$f(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (u-x)^{\nu-1} f^{(\nu)}(u)\, du, \qquad \nu > 0,$$

where $f^{(\nu)}$ means fractional derivative of f of order ν, has been taken as a basis to build functional calculi for $\mathbf{BV}_\nu$ functions, or $AC^{(\nu)}$ absolutely continuous functions in particular (see Sect. 3), looking for applications to approximation processes. The way to proceed consists of substituting the function $R_u^\nu(x) := (u-x)_+^\nu$ in the integral by its (bounded) operator version $R_u^\nu(A) := (u-A)_+^\nu$, although to have defined the objects $R_u^\nu(A)$—which may be properly called *Bochner–Riesz* operators—it is required a previous calculus like the L^∞ one corresponding to a spectral decomposition for A. So, the method is restricted to operators under this assumption ([BNT]). We show in this paper that an $AC^{(\nu)}$-calculus is possible without appealing to spectral decompositions, for holomorphic semigroups with polynomial growth on vertical lines, and that this is the concrete calculus that we need in order to provide a solution to the considerations that, suggested by the Fourier transform approach, we have done before.

Our main theorem is Theorem 4.1 where a functional calculus Φ is given for semigroups $a^z = e^{-zA}$ such that $\sup_{\Re z \geqslant \varepsilon} \|a^z\|/|z|^\mu < \infty$ for some $\mu \geqslant 0$, every $\varepsilon > 0$ and smooth functions with compact support as operating functions. This calculus is expressed via an integral formula involving Weyl fractional derivatives and an operator-valued kernel $G^\nu(u)$ which is the inverse Laplace transform of the vector function $a^z/z^{\nu+1}$, for $\nu > \mu$, and which on the other hand coincides with $R_u^\nu(A)$ too. As a matter of fact, Φ can be approximated by the holomorphic calculus in a precise sense which is explained in the statement.

The proof of the theorem is based on an analysis of consequences derived from the representation of the function $a^{1+z}/(\varepsilon+z)^{\nu+1}$, for $\varepsilon > 0$, on the half plane $\Re z > 0$ as the Poisson integral of its boundary values. We obtain estimates independent of ε when $\varepsilon \to 0^+$ so that we can “attain” the boundary and get the final formula. The idea of using the Poisson integral representation has been taken from [deL1] where holomorphic semigroups bounded on all of $\Re z > 0$ and usual derivatives of first order are considered. Once we started to write the final version of this paper we became aware of [deL2] where the same kernel $G^\nu(u)$ is introduced for $\nu = 1, 2$ as part of a characterization of the well-boundedness of A (this corresponds to our $AC^{(1)}$ case, see Sect. 6). Indeed, the kernel $G^\nu(u)$ also appears in [deL1], under a different form, for $\nu = 1$. The approach followed in [deL2] is quite different from ours.

At this point let us recall that one of the motivations to study functional calculi is that they are also intended sometimes as an equivalent way of describing (possibly unbounded) operators which either admit spectral decompositions or generate strongly continuous groups or semigroups of bounded operators. An important example of this is that of well-bounded operators, defined on general Banach spaces [Do, BBG]. The results of [deL2] quoted above correspond to regarding well-boundedness in terms of generation of holomorphic semigroups, whereas a well-bounded operator T on $[0, \infty)$ is originally defined as possessing a *decomposition of the identity* $E(u)$, $u \geqslant 0$, so that $f(T) = \int_0^\infty f'(u)\, E(u)\, du$, formally, for every $f \in AC^{(1)}$ (see [Do, BBG, deL2]). Both of these interpretations are generalized in our work, the kernel $G^{\nu}(u)$ playing for $AC^{(\nu+1)}$, $\nu>0$, a similar role to the decomposition E for $AC^{(1)}$. See Section 7.3 and Section 4 below.

The expression of the kernel $G^{\nu}(u)$ as a vector integral in the defining formula of Φ enables us to extend the calculus to another class $AC^{(\nu)}_{2,1}$ of absolutely continuous functions which is related to Herz spaces [He, Fl]. So we need an "ad hoc" theory of absolutely continuous functions of higher fractional order that we develop in Sections 2 and 3. Section 2 is devoted to preliminaries and we adapt there the Weyl fractional derivation and integration setting to a formulation suitable for our purposes. In particular we present a Leibniz formula for fractional derivatives of higher order of the product of two functions which seems to be new and interesting on its own. Usually, the extensions of Leibniz's rule to the different types of fractional derivation that occur in the literature fail to be symmetric, or need to handle infinite differentiation of the involved functions, or both of them. On the contrary, the formula given here (in Proposition 2.5 below) is symmetric and only requires derivation up to the proper order. We would like to emphasize the role played by the Gauss hypergeometric function in the construction of the formula. In Section 3, the theory of $AC^{(\nu)}$ and $AC^{(\nu)}_{2,1}$ spaces is exposed. Among other things we show, using the aforementioned Leibniz formula, that these are (complete) normed algebras for pointwise multiplication.

Sections 4, 5, and 6 are devoted to the main theorem. In Section 4 we give the statement and its first consequences or corollaries. Section 5 contains the proof, and Section 6 contains improvements of results in Section 4, Section 5, in the case that the semigroup is assumed to be *subhomogeneous*. It is now that the $AC^{(\nu)}_{2,1}$ spaces appear in connection with our calculus.

Finally, in Section 7 we include examples and results which are useful to test the accuracy of the method, as well as some applications or comments related to multipliers or approximation. Incidentally, it turns out that, as a consequence of results in [E], the functional calculus constructed in [BEJ] and [EJ] may be applied to operators which generate semigroups satisfying the growth assumptions considered in this paper. In that respect, see also [D1] and [D2, pp. 180,181], as well as [Du] where a Hörmander multiplier theorem for $L^p(G)$, $1<p<\infty$ and G a stratified Lie group, is proved.

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2. PRELIMINARIES

Functional Calculus of Holomorphic Functions

Let A be a closed, densely defined operator on a Banach space X. Suppose that the spectrum $\sigma(A)$ of A is not the whole plane $\mathbf{C}$. If A is a bounded operator then to any function f which is holomorphic in a bounded domain Ω containing $\sigma(A)$ and continuous on $\bar{\Omega}$ there corresponds a bounded operator on X defined by

$$f(A) = \int_{\partial\Omega} f(z)(z-A)^{-1}\,dz.$$

If $\sigma(A)$ is an unbounded set then we have to assume that f is holomorphic at $z=\infty$. This means by definition that f is holomorphic outside a compact set and there exists the finite $\lim_{|z|\to\infty} f(z)$. In this case $\partial\Omega$ is also compact and we put

$$f(A) = f(\infty) + \int_{\partial\Omega} f(z)(z-A)^{-1}\,dz. \tag{2.1}$$

The operator $f(A)$ does not depend on a particular choice of Ω. Note that the correspondence $f \to f(A)$ is linear, multiplicative and $f(A)A \subset Af(A) = g(A)$ when $g(z) = zf(z)$ [HP, p. 199].

Fractional Derivation

Let $\mathscr{E}$ be the class of C^∞ functions on $[0, \infty)$ that satisfy

$$\sup_{x\in[0,\infty)} \left| x^m \frac{d^n}{dx^n} f(x) \right| < \infty$$

for any $m, n = 0, 1, 2, \ldots$. For a function f in $\mathscr{E}$ the *Weyl fractional integral* $W^{-\nu}f$ of order $\nu > 0$ is defined by

$$W^{-\nu}f(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} f(t)\,dt,$$

for $x > 0$ (see [SKM]).

It is clear that $W^{-\nu}f \in \mathscr{E}$. Moreover the map $W^{-\nu}: \mathscr{E} \to \mathscr{E}$ is one-to-one and onto. The inverse map W^ν is called the *Weyl fractional derivative* of order ν. If ν is a natural number then W^ν coincides with the usual derivative $(-1)^\nu (d^\nu/dx^\nu)$. Let us define W^0 to be the identity operator. Then $W^\mu W^\nu = W^{\mu+\nu}$ for any $\mu, \nu \in \mathbf{R}$.

Typical examples of functions in $\mathscr{E}$ are $(1+x)^\delta e^{-\varepsilon x}$, $\delta \in \mathbf{R}$, $\varepsilon > 0$ and smooth functions on $[0, \infty)$ with compact support. In the former case $W^\nu f$

is again a function with compact support since $W^{\nu}f(x)=(-1)^n\ (d^n/dx^n)\ W^{-n+\nu}f(x)$ provided that $n\geqslant \nu$.

The function $W^{\nu}f$ can also be expressed in terms of difference operators Δ_t^n, defined by $\Delta_t^1 f(x)=f(x)-f(x+t)$ and $\Delta_t^{k+1}=\Delta_t^1\Delta_t^k$ for $k=1, 2, \ldots$.

PROPOSITION 2.1 (Marchaud's formula). *If $n>\nu$ then there exists a constant $c_{n,\nu}>0$ such that*

$$W^{\nu}f(x)=c_{n,\nu}\int_0^{\infty} t^{-\nu-1}\,\Delta_t^n f(x)\,dt$$

for any $f\in\mathscr{E}$.

Since our proof of the Marchaud's formula seems to be simpler than the others known (cf. [GT, SKM]) we present it here.

Proof. Integration by parts gives

$$\Delta_t^n f(x)=\int_0^{\infty} t^{n-1}\varphi_n\left(\frac{y-x}{t}\right) W^n f(y)\,dy,$$

where φ_1 is the characteristic function of the interval $[0, 1]$ and

$$\varphi_{k+1}(x)=\int_{x-1}^{x} \varphi_k(s)\,ds, \qquad k=1, 2, \ldots .$$

Therefore

$$\begin{aligned}\int_0^{\infty} t^{-\nu-1}\,\Delta_t^n f(x)\,dt &=\int_0^{\infty}\left(\int_0^{\infty} t^{n-\nu-2}\varphi_n\left(\frac{y-x}{t}\right)dt\right) W^n f(y)\,dy\\ &=c\int_x^{\infty}(y-x)^{n-\nu-1}\,W^n f(y)\,dy\\ &=c\Gamma(n-\nu)\,W^{\nu}f(x)\end{aligned}$$

with

$$c=\int_0^{\infty} t^{\nu-n}\varphi_n(t)\,dt>0. \quad \blacksquare$$

Remark. Exact computation of the constant $c_{n,\nu}$ in Marchaud's formula gives

$$\frac{1}{c_{n,\nu}}=\Gamma(-\nu)\sum_{k=0}^{n}\binom{n}{k}(-1)^k\,k^{\nu},$$

when ν is not a natural number. Indeed, as we have shown in the beginning of the proof, for $f \in \mathscr{E}$ we have

$$\Delta_1^n f(0) = (-1)^n \int_0^\infty \varphi_n(t) \frac{d^n f(t)}{dt^n} dt.$$

Since $\operatorname{supp} \varphi_n \subset [0, n]$, taking any function $f \in \mathscr{E}$ such that $f(t) = t^\nu$ on $[0, n]$, we get

$$\begin{aligned}
\frac{1}{c_{n,\nu}} &= \Gamma(n-\nu) \int_0^\infty t^{\nu-n} \varphi_n(t)\, dt \\
&= (-1)^n \frac{\Gamma(n-\nu)}{\nu(\nu-1)\cdots(\nu-n+1)} \Delta_1^n f(0) \\
&= \Gamma(-\nu) \sum_{k=0}^{n} \binom{n}{k} (-1)^k k^\nu.
\end{aligned}$$

If ν is a natural number, an application of de l'Hôpital rule to $\lim_{\mu \to \nu} 1/c_{n,\mu}$ gives

$$\frac{1}{c_{n,\nu}} = \frac{(-1)^{\nu+1}}{\nu!} \sum_{k=1}^{n} \binom{n}{k} (-1)^k k^\nu \ln k.$$

Leibniz Formula for Fractional Derivation

For $t, u > 0$ and $\nu > -1$ define on $(-\infty, \infty)$ a function $\varphi_{t,u}^\nu$ in the following way. When $x < t < u$ let

$$\begin{aligned}
\varphi_{t,u}^\nu(x) &= \frac{1}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \binom{\nu}{k}^2 (t-x)^k (u-x)^{\nu-k} \\
&= \frac{(u-x)^\nu}{\Gamma(\nu+1)} {}_2F_1\left(-\nu, -\nu; 1; \frac{t-x}{u-x}\right),
\end{aligned} \tag{2.2}$$

where ${}_2F_1$ is the Gaussian hypergeometric function. Put $\varphi_{t,u}^\nu(x) = \varphi_{u,t}^\nu(x)$ when $x < u < t$ and let $\varphi_{t,u}^\nu(x) = 0$ in all other cases.

LEMMA 2.2. *Assume $t < u$. The function $\varphi_{t,u}^\nu$ is differentiable for $x < t$ and*

$$\int_x^t (s-x)^\nu (\varphi_{t,u}^\nu)'(s)\, ds = \frac{-1}{\Gamma(\nu+1)} (t-x)^\nu [(u-x)^\nu - (u-t)^\nu]. \tag{2.3}$$

Its derivative $(\varphi^{\nu}_{t,u})'$ *is nonnegative when* $-1<\nu\leqslant 0$ *and nonpositive when* $\nu\geqslant 0$. *For* $\nu>-1/2$ *it satisfies*

$$|(\varphi^{\nu}_{t,u})'(x)|\leqslant c_\nu(u-x)^{\nu-1}. \tag{2.4}$$

Proof. We deduce directly from (2.2) that on $(-\infty,t)$ the function $\varphi^{\nu}_{t,u}$ is increasing when $-1<\nu\leqslant 0$ and decreasing when $\nu\geqslant 0$. By derivation we obtain

$$\begin{aligned}(\varphi^{\nu}_{t,u})'(x)&=-\frac{\nu+1}{\Gamma(\nu+1)}\sum_{k=0}^{\infty}\binom{\nu}{k+1}\binom{\nu}{k}(t-x)^k(u-x)^{\nu-k-1}\\&=-\frac{\nu(\nu+1)}{\Gamma(\nu+1)}\,{}_2F_1\left(-\nu,1-\nu;2;\frac{t-x}{u-x}\right)(u-x)^{\nu-1}.\end{aligned}$$

When $\nu>-1/2$, the function ${}_2F_1(-\nu,1-\nu;2;\cdot)$ is bounded on $[0,1]$, which implies (2.4).

Denote the left hand side of (2.3) by I. Then

$$-\Gamma(\nu+1)I=\sum_{k=1}^{\infty}\binom{\nu+1}{k}\binom{\nu}{k}k\int_x^t(s-x)^\nu(t-s)^{k-1}(u-s)^{\nu-k}\,ds.$$

Since

$$(u-s)^{\nu-k}=\sum_{m=0}^{\infty}\binom{\nu-k}{m}(-1)^m(s-x)^m(u-x)^{\nu-k-m}$$

and

$$\int_x^t(s-x)^{\nu+m}(t-s)^{k-1}\,ds=\frac{\Gamma(\nu+m+1)\,\Gamma(k)}{\Gamma(\nu+m+k+1)}(t-x)^{\nu+m+k},$$

we get

$$-\Gamma(\nu+1)I=\sum_{n=1}^{\infty}c_n(t-x)^{\nu+n}(u-x)^{\nu-n},$$

with

$$\begin{aligned}c_n&=\sum_{\substack{k+m=n\\k\geqslant 1}}(-1)^m\binom{\nu+1}{k}\binom{\nu}{k}\binom{\nu-k}{m}\frac{\Gamma(\nu+m+1)(k+1)}{\Gamma(\nu+m+k+1)}\\&=\binom{\nu}{n}\binom{\nu+n}{n}^{-1}\sum_{\substack{k+m=n\\k\geqslant 1}}\binom{\nu+1}{k}\binom{-\nu-1}{m}.\end{aligned}$$

Observe that by the Vandermonde's convolution formula the last sum above is $\binom{0}{n}-\binom{-\nu-1}{n}$. Therefore

$$c_n = -\binom{\nu}{n}\binom{\nu+n}{n}^{-1}\binom{-\nu-1}{n} = -(-1)^n\binom{\nu}{n},$$

and

$$\begin{aligned}-\Gamma(\nu+1)\,I &= -\sum_{n=1}^{\infty}\binom{\nu}{n}(-1)^n\,(t-x)^{\nu+n}\,(u-x)^{\nu-n}\\ &= (t-x)^{\nu}\,[(u-x)^{\nu}-(u-t)^{\nu}]. \quad \blacksquare\end{aligned}$$

COROLLARY 2.3. *For $\nu>0$ and $0\leqslant x<\min\{t,u\}$ the following formula holds*

$$\frac{1}{\Gamma(\nu)}\int_x^{\infty}(s-x)^{\nu-1}\,\varphi_{t,u}^{\nu}(s)\,ds = \frac{(t-x)^{\nu}}{\Gamma(\nu+1)}\,\frac{(u-x)^{\nu}}{\Gamma(\nu+1)}. \tag{2.5}$$

When $\nu\leqslant -1/2$, the function $(\varphi_{t,u}^{\nu})'$ becomes unbounded in t and u near the diagonal. We have however:

LEMMA 2.4. *For $\nu>-1$ there exists a constant c_ν such that*

$$\int_x^y\int_x^y(\varphi_{t,u}^{\nu})'\,(x)\,dt\,du = c_\nu(y-x)^{\nu+1} \qquad \textit{for any} \quad x<y.$$

Proof. If I denotes the integral, then $I=\lim_{\varepsilon\to 0+} 2I_\varepsilon$, where

$$I_\varepsilon = \iint_{\substack{x<t<u<y\\ u-t>\varepsilon}}(\varphi_{t,u}^{\nu})'\,(x)\,dt\,du.$$

Since $(\varphi_{t,u}^{\nu})'\,(x) = -(\partial/\partial t)\,\varphi_{t,u}^{\nu}(x) - (\partial/\partial u)\,\varphi_{t,u}^{\nu}(x)$, we get

$$\begin{aligned}I_\varepsilon &= -\int_{x+\varepsilon}^{y}\int_x^{u-\varepsilon}\frac{\partial}{\partial t}\,\varphi_{t,u}^{\nu}(x)\,dt\,du - \int_x^{y-\varepsilon}\int_{t+\varepsilon}^{y}\frac{\partial}{\partial u}\,\varphi_{t,u}^{\nu}(x)\,du\,dt\\ &= \int_{x+\varepsilon}^{y}\varphi_{x,u}^{\nu}(x)\,du - \int_x^{y-\varepsilon}\varphi_{t,y}^{\nu}(x)\,dt\\ &= \frac{1}{\Gamma(\nu+1)}\Bigg(\frac{(y-x)^{\nu+1}-\varepsilon^{\nu+1}}{\nu+1}\\ &\quad -\sum_{k=0}^{\infty}\binom{\nu}{k}^2\frac{1}{k+1}\left(\frac{y-x-\varepsilon}{y-x}\right)^{k+1}(y-x)^{\nu+1}\Bigg).\end{aligned}$$

Thus

$$I = \frac{2}{\Gamma(\nu+1)} \left(\frac{1}{\nu+1} - {}_2F_1(-\nu, -\nu; 2; 1) \right) (y-x)^{\nu+1}$$

$$= \frac{2}{\Gamma(\nu+1)} \left(\frac{1}{\nu+1} - \frac{\Gamma(2\nu+2)}{\Gamma(\nu+2)^2} \right) (y-x)^{\nu+1}. \quad \blacksquare$$

We are now in a position to establish a formula for the fractional derivative of a product of functions.

PROPOSITION 2.5 (Leibniz formula). *Assume that $f, g \in \mathscr{E}$ and $\nu > 0$. Then*

$$W^\nu(f \cdot g)(x) = W^\nu f(x)\, g(x) + f(x)\, W^\nu g(x)$$
$$- \int_x^\infty \int_x^\infty \frac{d\varphi_{t,u}^{\nu-1}(x)}{dx}\, W^\nu f(t)\, W^\nu g(u)\, dt\, du$$

and the double integral is absolutely convergent.

Proof. We know from Lemma 2.2 that $(\varphi_{t,u}^{\nu-1})'$ is of constant sign. Thus the absolute convergence of the double integral follows immediately from Lemma 2.4.

Denote by $h(x)$ the right hand side of the formula. We have to show that $W^{-\nu}h(x) = f(x)\, g(x)$. To do this write $f = W^{-\nu}(W^\nu f)$ and $g = W^{-\nu}(W^\nu g)$. Then

$$W^{-\nu}(W^\nu f \cdot g + f \cdot W^\nu g)(x) = \frac{1}{\Gamma(\nu)^2} \int_x^\infty \int_x^\infty \xi(t,u)\, W^\nu f(t)\, W^\nu g(u)\, dt\, du,$$

where

$$\xi(t,u) = \begin{cases} (t-x)^{\nu-1}\,(u-t)^{\nu-1} & \text{for } \ t \leqslant u, \\ (u-x)^{\nu-1}\,(t-u)^{\nu-1} & \text{for } \ \mathrm{u} \leqslant \mathrm{t}. \end{cases}$$

An application of $W^{-\nu}$ to the third factor of h by Fubini's theorem gives

$$-\frac{1}{\Gamma(\nu)} \int_x^\infty \int_x^\infty \left(\int_x^{\min\{t,u\}} (s-x)^{\nu-1} \frac{d\varphi_{t,u}^{\nu-1}(s)}{ds}\, ds \right) W^\nu f(t)\, W^\nu g(u)\, dt\, du$$

and since the integral in brackets by the formula (2.3) of the Lemma 2.2 becomes equal to $\Gamma(\nu)^{-1}[\xi(t,u)-(t-x)^{\nu-1}(u-x)^{\nu-1}]$ we get

$$W^{-\nu}h(x)=\frac{1}{\Gamma(\nu)^2}\int_x^\infty\int_x^\infty(t-x)^{\nu-1}(u-x)^{\nu-1}W^\nu f(t)\,W^\nu g(u)\,dt\,du$$
$$=f(x)\cdot g(x). \quad \blacksquare$$

Remark. The formula just established reduces to the usual Leibniz's rule when ν is a natural number. Since

$$(\varphi_{t,u}^{\nu-1})'(x)=-\frac{\nu}{\Gamma(\nu)}\sum_{k=1}^{\nu-1}\binom{\nu-1}{k}\binom{\nu-1}{k-1}(t-x)^{k-1}(u-x)^{\nu-k-1}$$

for $x<\max\{t,u\}$, we have

$$-\int_x^\infty\int_x^\infty(\varphi_{t,u}^{\nu-1})'(x)\,W^\nu f(t)\,W^\nu g(u)\,dt\,du=\sum_{k=1}^{\nu-1}\binom{\nu}{k}W^{\nu-k}f(x)\,W^k g(x)$$

and so $W^\nu(fg)=\sum_{k=0}^{\nu}\binom{\nu}{k}W^{\nu-k}f\,W^k g$.

There are many versions of Leibniz's rule concerned with fractional differentiation [SKM, pp. 316, 317]. Perhaps the best known formula of this type is $D_{a^+}^\nu=\sum_{k=0}^\infty\binom{\nu}{k}D_{a^+}^{\nu-k}f\,D_{a^+}{}^\nu g$, valid for analytic functions f, g, where $D_{a^+}^\nu$ denotes the Riemann–Liouville derivative [SKM, p. 280]. A similar equality holds if we put W^ν instead of $D_{a^+}^\nu$, for suitable f, g making the corresponding series convergent (for, it is enough to use (6.9) of [MR, p. 247] and the analytic continuation principle since $\nu\to W^\nu f$ is an entire function for every $f\in\mathscr{E}$).

Observe that the above identity is not symmetric in f, g; as well, it requires the use of infinite differentiability. Among the results that, in one way or another, try to overcome these restrictions it should be mentioned here the expression obtained by Gaer and Rubel on the basis of the Mittag–Leffler summation [GR, p. 191]. Such a formula is symmetric but also applies exclusively to analytic functions and, moreover, it does not reduce to the Leibniz's rule even for small order of derivation. The Gaer–Rubel's approach is particularly interesting in connection with our Proposition 2.5 because $g^{(\nu)}(-x)=W^\nu f(x)$ $(x,\nu>0)$, for every analytic, rapidly decreasing at infinity function f on $\mathbf{R}$, where $g(x)=f(-x)$ and $g^{(\nu)}$ is now Gaer–Rubel's derivative [GR, p. 199].

Finally, let us notice that the formula that we present in Proposition 2.5 resembles slightly the Leibniz's formula with a remainder referred to in [SKM, p. 316]. There is some precedent of the use of the Gauss hypergeometric function as an integral kernel in the introduction of certain integral operators related to fractional calculus [SKM, pp. 438, 439]. Formula (2.5) in our Corollary 2.3 is (4.4) of [MS].

3. SPACES OF ABSOLUTELY CONTINUOUS FUNCTIONS

For $\nu > 0$ denote by $AC^{(\nu)}$ the Banach space obtained as the completion of the space $\mathscr{E}$ in the norm

$$\|f\|_{(\nu)} = \frac{1}{\Gamma(\nu)} \int_0^\infty |W^\nu f(x)|\, x^{\nu-1}\, dx.$$

Note that $f \in AC^{(\nu)}$ if and only if there exists a measurable function g on $[0, \infty)$ such that $\int_0^\infty |g(x)|\, x^{\nu-1}\, dx < \infty$ and

$$f(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1}\, g(t)\, dt. \tag{3.1}$$

The function g is unique (up to a set of measure 0), and will be called the *derivative of f of order ν* and denoted by $W^\nu f$. Although the integral (3.1), when $0 < \nu < 1$, may diverge for some values of x, the double integral

$$\int_0^r \int_x^\infty (t-x)^{\nu-1}\, |g(t)|\, dt\, dx$$

is finite for every $r > 0$. It follows from the Fubini theorem that (3.1) is (absolutely) convergent for almost all x and defines a locally integrable function.

Motivated by the fact that $AC^{(1)}$ consists exactly of those absolutely continuous functions of bounded variation on $[0, \infty)$ which tends to zero at infinity, the functions in $AC^{(\nu)}$ will be called *absolutely continuous of order ν*.

Spaces of that type have been considered in the literature in relation with functional calculi, multipliers, and approximation theory [BNT, T, and references therein]. Some general properties of $AC^{(\nu)}$, as closed subspaces (subalgebras) of corresponding spaces $\mathbf{BV}_\nu$, can be found in [T]. In this section, we present those results of [T] in a slightly stronger and simpler form, and we give some others as well as for a wider class of functions than $AC^{(\nu)}$, formed by the functions f for which

$$\int_0^\infty \left[\int_y^{2y} \left| W^\nu f(x)\, x^\nu \right|^2 \frac{dx}{x} \right]^{1/2} \frac{dy}{y} < \infty.$$

Significant examples of functions in $AC^{(\nu)}$ are the Bochner–Riesz functions

$$R_\lambda^\theta(x) = \frac{1}{\Gamma(\theta+1)} (\lambda - x)_+^\theta \tag{3.2}$$

with $\theta > \nu - 1$ and $\lambda > 0$. Indeed, if $\mu > 0$ then

$$\begin{aligned} W^{-\mu} R_\lambda^\theta(x) &= \frac{1}{\Gamma(\mu)\,\Gamma(\theta+1)} \int_x^\infty (s-x)^{\mu-1} (\lambda - s)_+^\theta \, ds \\ &= \frac{(\lambda - x)_+^{\theta+\mu}}{\Gamma(\mu)\,\Gamma(\theta+1)} \int_0^1 s^{\mu-1}(1-s)^\theta \, ds = R_\lambda^{\theta+\mu}(x), \end{aligned}$$

it follows that $W^\nu R_\lambda^\theta(x) = R_\lambda^{\theta-\nu}(x)$ is integrable whenever $\nu < \theta + 1$ and $\|R_\lambda^\theta\|_{(\nu)} = R_\lambda^\theta(0) = \lambda^\theta / \Gamma(\theta+1)$. Let us also remark that in Corollary 2.3 we have actually shown that $W^\theta(R_t^\theta \cdot R_u^\theta) = \varphi_{t,u}^\theta$.

PROPOSITION 3.1. *Assume $f \in AC^{(\nu)}$. Then*

(i) $\|f\|_{(\mu)} \leqslant \|f\|_{(\nu)}$ *whenever* $\mu \leqslant \nu$,

(ii) *if* $\nu \geqslant 1$ *then the function* $x^{\nu-1} W^{\nu-1} f(x)$ *is bounded, absolutely continuous, tends to zero at infinity and*

$$\sup_{x \in [0,\infty)} |x^{\nu-1} W^{\nu-1} f(x)| \leqslant \Gamma(\nu)\, \|f\|_{(\nu)}.$$

In particular $\|f\|_\infty \leqslant \|f\|_{(\nu)}$.

Proof. We have

$$\begin{aligned} \|f\|_{(\mu)} &= \frac{1}{\Gamma(\mu)} \int_0^\infty \left| \frac{1}{\Gamma(\nu-\mu)} \int_x^\infty (t-x)^{\nu-\mu-1} \, W^\nu f(t)\, dt \right| x^{\mu-1} \, dx \\ &\leqslant \frac{1}{\Gamma(\mu)\,\Gamma(\nu-\mu)} \int_0^\infty \int_0^t (t-x)^{\nu-\mu-1} x^{\mu-1} \, dx \, |W^\nu f(t)| \, dt \\ &= \frac{B(\mu, \nu-\mu)}{\Gamma(\mu)\Gamma(\nu-\mu)} \int_0^\infty |W^\nu f(t)| \, t^{\nu-1} \, dt = \|f\|_{(\nu)}, \end{aligned}$$

which proves (i). Part (ii) follows immediately from the formula

$$x^{\nu-1} W^{\nu-1} f(x) = \int_x^\infty W^\nu f(t) x^{\nu-1} \, dt. \quad \blacksquare$$

COROLLARY 3.2. *Suppose that n is a natural number. The necessary and sufficient conditions in order that a function f on $[0, \infty)$ belongs to $AC^{(n)}$ are that f has all derivatives up to order $n-1$ satisfying*

$$\lim_{x \to \infty} x^k f^{(k)}(x) = 0 \qquad \text{for} \quad k = 0, 1, 2, ..., n-1,$$

and the $(n-1)$th derivative is absolutely continuous with

$$\int_0^\infty x^{n-1} \, |f^{(n)}(x)| \, dx < \infty.$$

COROLLARY 3.3. *The space $AC^{(\nu)}$ contains all the complex functions on $[0, \infty)$ which vanish at infinity and have holomorphic extension to a neighborhood of $[0, \infty) \cup \{\infty\}$. The set of these functions is dense in $AC^{(\nu)}$.*

Proof. The first part follows since the function $z \to f(1/z)$ is holomorphic at 0 and so $f^{(n)}(x) = O(x^{-n-1})$ at ∞ for any $n = 1, 2, ...$. To see the second part observe that even the set of functions $x \to p(1/(1+x))$, p a polynomial without constant term, is dense in $AC^{(\nu)}$. ∎

Remark. The main reason we prefer to use $AC^{(\nu)}$ spaces instead of $\mathbf{BV}_\nu$ mentioned earlier is Corollary 3.3. Recall that, up to constants, $\mathbf{BV}_\nu$ consists of all functions f for which $W^\nu f$ is a complex (not necessarily absolutely continuous) measure m on $[0, \infty)$ satisfying $\int_0^\infty x^{\nu-1} \, |dm(x)| < \infty$. Observe that if $\mu < \nu$ then

$$W^\mu f(x) = \frac{1}{\Gamma(\nu - \mu)} \int_x^\infty (t-x)^{\nu-\mu-1} \, dm(t)$$

is already a function. Therefore $f \in \bigcap_{\mu < \nu} AC^{(\mu)}$. From this point of view the difference between $AC^{(\nu)}$ and $\mathbf{BV}_\nu$ is unessential.

The fact that $AC^{(\nu)}$ is closed under multiplication may be proved as in [T], but we show here that this is a direct consequence of the Leibniz formula.

PROPOSITION 3.4. *If $\nu \geqslant 1$ then $AC^{(\nu)}$ is a Banach algebra under pointwise multiplication.*

Proof. Let $f, g \in AC^{(\nu)}$. We are going to show that $\|f \cdot g\|_{(\nu)} \leqslant \|f\|_{(\nu)} \|g\|_{(\nu)}$. To compute $W^\nu(f \cdot g)$ we use the Leibniz formula of Proposition 2.5 (although originally it was formulated only for functions in $\mathscr{E}$ it is applicable also here because all the integrals are absolutely convergent). Since $\varphi_{t,u}^{\nu-1}(x)$ is a decreasing function of x we have

$$-\frac{d\varphi_{t,u}^{\nu-1}(x)}{dx} \geqslant 0 \qquad \text{for} \quad 0 < x < \min\{t, u\}$$

and so

$$|W^\nu(f \cdot g)(x)| \leqslant |W^\nu f(x)| \, |g(x)| + |f(x)| \, |W^\nu g(x)|$$
$$- \int_x^\infty \int_x^\infty \frac{d\varphi_{t,u}^{\nu-1}(x)}{dx} \, |W^\nu f(t)| \, |W^\nu g(u)| \, dt \, du.$$

If for $h \in AC^{(\nu)}$ we denote by $\tilde{h}$ the function in $AC^{(\nu)}$

$$\tilde{h}(x) = \frac{1}{\Gamma(\nu)} \int_x^\infty (t-x)^{\nu-1} \, |W^\nu h(t)| \, dt,$$

then since $|h(x)| \leqslant \tilde{h}(x)$ on $[0, \infty)$ and since $\|h\|_{(\nu)} = \|\tilde{h}\|_{(\nu)} = \tilde{h}(0)$ we get $|W^\nu(f \cdot g)(x)| \leqslant W^\nu(\tilde{f} \cdot \tilde{g})(x)$ and consequently

$$\|f \cdot g\|_{(\nu)} \leqslant \|\tilde{f} \cdot \tilde{g}\|_{(\nu)} = (\tilde{f} \cdot \tilde{g})(0) = \|f\|_{(\nu)} \|g\|_{(\nu)}. \quad \blacksquare$$

Remark. Proposition 3.4 is false for $0 < \nu < 1$. Indeed, if $2\mu < \nu - 1 < \mu$, then $R_\lambda^\mu \in AC^{(\nu)}$ but $R_\lambda^{2\mu}$ not.

The spaces $AC^{(\nu)}$ are invariant under potential changes of variable.

PROPOSITION 3.5. *Let $\alpha > 0$. The transformation $x \to x^\alpha$ of $[0, \infty)$ induces an isomorphism $f(x) \to f(x^\alpha)$ of the space $AC^{(\nu)}$.*

Proof. First of all observe that the function

$$\varphi(x) = (1 - x^\alpha)_+^{\nu-1} - \alpha^{\nu-1}(1-x)_+^{\nu-1}$$

belongs to $AC^{(\nu)}$. Indeed, we can write $\varphi(x) = \varphi_0(x)(1-x)_+^\nu$, where φ_0 is a smooth function (even real analytic) on $(0, \infty)$ such that $x^m \varphi_0^{(m+1)}(x)$ is integrable near 0 for any m, for we have $x^m (d^{m+1}/dx^{m+1})(1-x^\alpha)^{\nu-1} = O(x^{\alpha-1})$ as $x \to 0^+$. It is an immediate consequence of Corollary 3.2 that $\varphi \in AC^{(n)}$ whenever $n < \nu + 1$. But $AC^{(n)} \subset AC^{(\nu)}$ if $n \geqslant \nu$.

Now let $f \in \mathscr{E}$ and put $g(x) = f(x^\alpha)$, $x \geqslant 0$. Let also $\tilde{\varphi}$ denote the W^ν derivative of the function φ above, i.e.

$$\varphi(x) = (1 - x^\alpha)_+^{\nu-1} - \alpha^{\nu-1}(1-x)_+^{\nu-1} = \frac{1}{\Gamma(\nu)} \int_x^1 (u-x)^{\nu-1} \, \tilde{\varphi}(u) \, du \qquad (3.3)$$

when $0<x<1$. We have then

$$g(x)=f(x^\alpha)=\frac{1}{\Gamma(\nu)}\int_{x^\alpha}^{\infty}(t-x^\alpha)^{\nu-1}\,W^\nu f(t)\,dt$$
$$=\frac{1}{\Gamma(\nu)}\int_{x}^{\infty}(t^\alpha-x^\alpha)^{\nu-1}\,t^{(1-\alpha)(\nu-1)}h(t)\,dt,$$

where we denoted by h the function $h(t)=\alpha t^{(\alpha-1)\nu}\,W^\nu f(t^\alpha)$. We deduce from (3.3) that

$$(t^\alpha-x^\alpha)^{\nu-1}\,t^{(1-\alpha)(\nu-1)}=\alpha^{\nu-1}(t-x)^{\nu-1}+\frac{1}{\Gamma(\nu)}\int_x^t(u-x)^{\nu-1}\,\tilde\varphi\left(\frac{u}{t}\right)\frac{du}{t}$$

whenever $0<x<t$. Thus

$$g(x)=\frac{\alpha^{\nu-1}}{\Gamma(\nu)}\int_x^\infty(t-x)^{\nu-1}\,h(t)\,dt$$
$$+\frac{1}{\Gamma(\nu)^2}\int_x^\infty(u-x)^{\nu-1}\int_u^\infty\tilde\varphi\left(\frac{u}{t}\right)h(t)\,\frac{dt}{t}\,du$$
$$=\frac{1}{\Gamma(\nu)}\int_x^\infty(u-x)^{\nu-1}\,\tilde g(u)\,du,$$

where

$$\tilde g(u)=\alpha^{\nu-1}h(u)+\frac{1}{\Gamma(\nu)}\int_u^\infty\tilde\varphi\left(\frac{u}{t}\right)h(t)\,\frac{dt}{t}.$$

This implies by the definition that $W^\nu g=\tilde g$ and we can estimate

$$\|g\|_{(\nu)}=\frac{1}{\Gamma(\nu)}\int_0^\infty|\,\tilde g(u)|\,u^{\nu-1}\,du$$
$$\leqslant\frac{\alpha^{\nu-1}}{\Gamma(\nu)}\int_0^\infty|h(u)|\,u^{\nu-1}\,du$$
$$+\frac{1}{\Gamma(\nu)^2}\int_0^\infty\int_0^t\left|\tilde\varphi\left(\frac{u}{t}\right)\right|u^{\nu-1}\,du\,|h(t)|\,\frac{dt}{t}$$
$$=\left(\alpha^{\nu-1}+\frac{1}{\Gamma(\nu)}\int_0^1|\tilde\varphi(u)|\,u^{\nu-1}\,du\right)\frac{1}{\Gamma(\nu)}\int_0^\infty|h(t)|\,t^{\nu-1}\,dt,$$
$$=(\alpha^{\nu-1}+\|\varphi\|_{(\nu)})\,\|f\|_{(\nu)}.$$

This means that the map $f(x) \to f(x^{\alpha})$ is continuous from $AC^{(\nu)}$ to $AC^{(\nu)}$. But since the inverse map $f(x) \to f(x^{1/\alpha})$ is also continuous, it is an isomorphism. ∎

We also need to discuss in this paper another family of spaces of absolutely continuous functions. For $\nu > 0$ we denote by $AC^{(\nu)}_{2,1}$ the completion of $\mathscr{E}$ in the norm

$$\|f\|_{(\nu),2,1} = \int_0^\infty \left[\int_y^{2y} |x^\nu W^\nu f(x)|^2 \frac{dx}{x} \right]^{1/2} \frac{dy}{y}.$$

This norm is equivalent to

$$\sum_{-\infty}^{\infty} 2^{k\nu} \|\chi_{[2^k, 2^{k+1}]} W^\nu f\|_2 \qquad \text{and to} \qquad \int_0^\infty \|\varphi W^\nu f_y\|_2 \frac{dy}{y},$$

where $f_y(x) = f(yx)$ and φ is any nontrivial bounded function with compact support in $(0, \infty)$. By $\|\cdot\|_2$ we denoted here the norm $\|g\|_2 = (\int_0^\infty |g(x)|^2 (dx/x)^{1/2}$.

Lemma 3.6. *Assume* $0 \leqslant \mu < \nu - 1/2$. *If* $f \in AC^{(\nu)}_{2,1}$ *then the function* $x^\mu W^\mu f(x)$ *is uniformly continuous, tends to zero at infinity and*

$$|x^\mu W^\mu f(x)| \leqslant C_{\mu,\nu} \|f\|_{(\nu),2,1}, \tag{3.4}$$

where $C_{\mu,\nu}$ *is a positive constant, independent of* f.

Proof. We have

$$x^\mu W^\mu f(x) = \frac{1}{\Gamma(\nu-\mu)} \int_x^\infty (t-x)^{\nu-\mu-1} x^\mu W^\nu f(t)\, dt.$$

If $2^{n-1} \leqslant x < 2^n$, then we can write this integral as $\int_x^{2^n} + \sum_{k=n}^\infty \int_{2^k}^{2^{k+1}}$. Applying the Schwarz inequality to each factor we get

$$|x^\mu W^\mu f(x)| \leqslant C \sum_{k=n}^{\infty} 2^{k\nu} \|\chi_{[2^{k-1}, 2^k]} W^\nu f\|_2 \leqslant C \|f\|_{(\nu),2,1}. \quad ∎$$

Note that Lemma 3.6 is not true when $\mu = \nu - 1/2$ and that $AC^{(1/2)}_{2,1}$ contains some discontinuous unbounded functions.

Proposition 3.7. (i) $AC^{(\nu)}_{2,1} \subset AC^{(\mu)}_{2,1}$ *for* $\nu > \mu > 1/2$,

(ii) $AC^{(\nu+1/2)} \subset AC^{(\mu)}_{2,1} \subset AC^{(\mu)}$ *for* $\nu > \mu > 0$.

Proof. Denote for simplicity $f_\mu(x)=x^\mu W^\mu f(x)$. Then since

$$f_\mu(x)=\frac{1}{\Gamma(\nu-\mu)}\int_0^\infty t^{\nu-\mu-1}x^\mu(x+t)^{-\nu} f_\nu(x+t)\,dt$$

and

$$x^{\mu-1/2}(x+t)^{-\nu+1/2}\chi_{[y,2y]}(x)\leqslant(2y)^{\mu-1/2}(y+t)^{-\nu+1/2}\chi_{[y+t,2y+2t]}(x+t),$$

the Minkowski integral inequality gives

$$\begin{aligned}\|f\|_{(\mu),2,1}&=\int_0^\infty y^{-1}\|\chi_{[y,2y]}f_\mu\|_2\,dy\\&\leqslant\frac{2^{\mu-1/2}}{\Gamma(\nu-\mu)}\int_0^\infty\int_0^\infty t^{\nu-\mu-1}y^{\mu-3/2}\\&\quad\times(y+t)^{-\nu+1/2}\|\chi_{[y+t,2y+2t]}f_\nu\|_2\,dt\,dy\\&=2^{\mu-1/2}\Gamma(\mu-1/2)\,\Gamma(\nu-1/2)^{-1}\|f\|_{(\nu),2,1}.\end{aligned}$$

To get (ii) write $f_\mu(x)$ as $\int_0^\infty t^{\nu-1/2}W^{\nu+1/2}f(t)\,g_t(x)\,dt$, where

$$g_t(x)=\frac{1}{\Gamma(\nu-\mu+1/2)}(t-x)_+^{\nu-\mu-1/2}x^\nu t^{-\nu+1/2}.$$

The Minkowski inequality again gives

$$\begin{aligned}\|f\|_{(\mu),2,1}&\leqslant\int_0^\infty t^{\nu-1/2}|W^{\nu+1/2}f(t)|\int_0^\infty y^{-1}\|\chi_{[y,2y]}g_t\|_2\,dy\,dt\\&=\int_0^\infty y^{-1}\|\chi_{[y,2y]}g_1\|_2\,dy\,\|f\|_{(\nu+1/2)}.\end{aligned}$$

On the other hand $\int_0^\infty \chi_{[y,2y]}(x)\,y^{-1}\,dy=\ln 2$ for any $x>0$. Thus

$$\begin{aligned}\|f\|_{(\mu)}&=\frac{1}{\Gamma(\mu)\ln 2}\int_0^\infty\int_y^{2y}|f_\mu(x)|\,x^{-1}\,dx\,y^{-1}\,dy\\&\leqslant\frac{1}{\Gamma(\mu)\sqrt{\ln 2}}\|f\|_{(\mu),2,1},\end{aligned}$$

by the Schwarz inequality. ∎

In addition to Proposition 3.7 note that the Minkowski and Schwarz inequalities also imply that

$$\begin{aligned} \| W^\nu f(x) x^\nu \|_2 &\leqslant \frac{1}{\ln 2} \, \| f \|_{(\nu), 2, 1}, \\ \| f \|_{(\nu), 2, 1} &\leqslant \sqrt{\ln 4} \, \| W^\nu f(x) \, x^\nu (1 + |\log x|) \|_2. \end{aligned} \tag{3.5}$$

PROPOSITION 3.8. *If $\nu > 1/2$ then $AC_{2,1}^{(\nu)}$ is closed under multiplication.*

Proof. It is enough to show that $\| f^2 \|_{(\nu), 2, 1} \leqslant C \, \| f \|^2_{(\nu), 2, 1}$ for every function $f \in \mathscr{E}$ and some constant C. Fix f and denote by $\tilde{f}$ the function $\tilde{f} = W^{-\nu}(|W^\nu f|)$. Then obviously $\tilde{f} \in AC_{2,1}^{(\nu)}$ and $\| \tilde{f} \|_{(\nu), 2, 1} = \| f \|_{(\nu), 2, 1}$. To estimate $|W^\nu(f^2)|$ we use the Leibniz formula and (2.4). We have, namely,

$$\begin{aligned} |W^\nu(f^2)(x)| &\leqslant 2 \, |f(x) \, W^\nu f(x)| \\ &\quad + C \int_x^\infty \int_x^u (u - x)^{\nu - 2} \, |W^\nu f(t) \, W^\nu f(u)| \, dt \, du. \end{aligned}$$

Take ε such that $1/2 < \varepsilon < \min\{1, \nu\}$. Then $(u - x)^{\varepsilon - 1} \leqslant (t - x)^{\varepsilon - 1}$, if $t \leqslant u$, so the last factor above is dominated by

$$\begin{aligned} & C \int_x^\infty (t - x)^{\varepsilon - 1} \, |W^\nu f(t)| \, dt \int_x^\infty (u - x)^{\nu - \varepsilon - 1} \, |W^\nu f(u)| \, du \\ & \quad \leqslant C' \, W^{\nu - \varepsilon} \tilde{f}(x) \, W^\varepsilon \tilde{f}(x). \end{aligned}$$

This implies that

$$|x^\nu W^\nu(f^2)(x)| \leqslant C_1 |x^\nu W^\nu f(x)| + C_2 |x^\varepsilon W^\varepsilon \tilde{f}(x)|,$$

where by (3.4)

$$C_1 = 2 \sup_{0 < x < \infty} |f(x)| \leqslant 2 C_{0, \nu} \, \| f \|_{(\nu), 2, 1},$$

$$C_2 = C' \sup_{0 < x < \infty} |x^{\nu - \varepsilon} W^{\nu - \varepsilon} \tilde{f}(x)| \leqslant C' C_{\nu - \varepsilon, \nu} \| f \|_{(\nu), 2, 1}.$$

Therefore

$$\| f^2 \|_{(\nu), 2, 1} \leqslant C_1 \, \| f \|_{(\nu), 2, 1} + C_2 \, \| \tilde{f} \|_{(\varepsilon), 2, 1} \leqslant C \, \| f \|^2_{(\nu), 2, 1}. \quad \blacksquare$$

We will make use of the analog to Proposition 3.5 later, in Section 7.

PROPOSITION 3.9. *For $\alpha > 0$ the transformation $x \to x^\alpha$ of $[0, \infty)$ induces an isomorphism $f(x) \to f(x^\alpha)$ of the space $AC_{2,1}^{(\nu)}$.*

Proof. In the same notations as in Proposition 3.5, for the function $g(x)=f(x^\alpha)$ we get

$$\|g\|_{(\nu),2,1}=\int_0^\infty \left[\int_y^{2y} |u^\nu \tilde{g}(u)|^2 \frac{du}{u}\right]^{1/2} \frac{dy}{y}$$
$$\leqslant (\alpha^{\nu-1}+\|\varphi\|_{(\nu)}) \int_0^\infty \left[\int_y^{2y} |t^\nu h(t)|^2 \frac{dt}{t}\right]^{1/2} \frac{dy}{y}$$

by the Minkowski inequality. But

$$\int_0^\infty \left[\int_y^{2y} |t^\nu h(t)|^2 \frac{dt}{t}\right]^{1/2} \frac{dy}{y}$$
$$=\alpha^{-1/2} \int_0^\infty \left[\int_y^{2^\alpha y} |x^\nu W^\nu f(x)|^2 \frac{dx}{x}\right]^{1/2} \frac{dy}{y}$$
$$\leqslant \alpha^{-1/2}(\alpha+1)\ \|f\|_{(\nu),2,1}.$$

Thus we conclude that the map $f(x)\to f(x^\alpha)$ is continuous from $AC^{(\nu)}_{2,1}$ into itself. ∎

Remark. The spaces $AC^{(\nu)}$ and $AC^{(\nu)}_{2,1}$ are particular examples of spaces $AC^{(\nu)}_{p,q}$, $1\leqslant p, q\leqslant\infty$, defined as the completion of $\mathscr{E}$ in the norm

$$\|f\|_{(\nu),p,q}=\left(\int_0^\infty \left(\int_y^{2y} |x^\nu W^\nu f(x)|^p \frac{dx}{x}\right)^{q/p} \frac{dy}{y}\right)^{1/q},$$

with the usual changes in the form of this norm when p or $q=\infty$. Spaces of that type have been frequently applied in the theory of Fourier multipliers, especially for $q=\infty$ (see for example [C, CGT1, CGT2, GT) and for $p=q=1$ [BNT, T]. Many properties of these spaces and possible embeddings can be shown by methods developed in [SKM] and [CGT2]. There is also an interesting connection between spaces $AC^{(\nu)}_{p,q}$ and the so-called Herz spaces $K^\nu_{p,q;1}$ [He, Fl].

4. THE MAIN RESULT

Let $\{a^z\}_{\Re z>0}$ be a holomorphic semigroup of operators on a Banach space X. Assume that it is strongly continuous at the origin, i.e.

$$\lim_{\substack{t\to 0\\ t>0}} \|a^t x - x\| = 0 \qquad \text{for every} \quad x\in X \tag{C_0}$$

and for some real α satisfies the following growth condition:

$$\sup_{\Re z \geqslant \varepsilon} \frac{\|a^z\|}{|z|^\nu} < \infty \text{ for every } \varepsilon > 0 \text{ and every } \nu > \alpha. \tag{G_α}$$

The lower bound of all such α will be called the *growth exponent* of the semigroup.

It is well known that the infinitesimal generator $-A$ of the semigroup $\{a^t\}_{t>0}$ is closed and densely defined (c.f. [HP, Theorem 10.5.3]). Furthermore (G_α) implies that the spectrum $\sigma(A)$ of A lies in $[0, \infty)$ ([HP, p. 457]). This is because $\sup_{\Re z \geqslant 1} |e^{-z\lambda}|/|z|^\nu$ must be finite for any λ in $\sigma(A)$. Note also that each of the subspaces $X_z = \{a^z x : x \in X\}$, $\Re z > 0$, is dense in X. Indeed, if $x^* \in X^*$ annihilates X_{z_0} then each holomorphic function $z \to x^*(a^z x)$, $x \in X$, must be zero when $\Re z > \Re z_0$, so it must be zero identically. In particular $x^*(x) = 0$ for any $x \in X$ by (C_0).

THEOREM 4.1. *Let $-A$ be the infinitesimal generator of a holomorphic (C_0) semigroup $\{a^z\}_{\Re z > 0}$ of operators on a Banach space X which satisfies (G_α). Fix $\nu > \alpha$ and for a smooth function f: $[0, \infty) \to \mathbf{C}$ with compact support define a bounded operator $\Phi(f)$ on X by*

$$\Phi(f) = \int_0^\infty W^{\nu+1} f(u)\, G^\nu(u)\, du, \tag{4.1}$$

where $W^{\nu+1} f$ denotes the Weyl fractional derivative of f of order $\nu+1$ and G^ν the operator valued kernel

$$G^\nu(u) = \frac{1}{2\pi i} \int_{\Re z = 1} \frac{a^z}{z^{\nu+1}}\, e^{uz}\, dz. \tag{4.2}$$

Then Φ defines a functional calculus for A. Namely,

(i) *Φ does not depend on ν, i.e.*

$$\int_0^\infty W^{\mu+1} f(u)\, G^\mu(u)\, du = \int_0^\infty W^{\nu+1} f(u)\, G^\nu(u)\, du$$

whenever $\mu, \nu > \alpha$.

(ii) *The correspondence $f \to \Phi(f)$ is linear, multiplicative and satisfies $\Phi(f) A \subset A\Phi(f) = \Phi(g)$ when $g(t) = t \cdot f(t)$.*

The functional calculus Φ is related to the holomorphic functional calculus, defined by (2.1), *in the following manner*:

(iii) *There exists a sequence of complex functions $f_1, f_2, \ldots$, each one holomorphic in a neighborhood of $[0, \infty) \cup \{\infty\}$, such that $f_n \to f$ uniformly on $[0, \infty)$ and $f_n(A)a^z \to \Phi(f)a^z$ in the operator norm for any z, $\Re z > 0$. In particular $f_n(A)\,y \to \Phi(f)\,y$ for every y in the dense subspace $X_0 = \bigcup_{\Re z > 0} \{a^z x : x \in X\}$ of X.*

We postpone the proof of the theorem to the next section.

Remarks. 1. It is clear that the operation Φ can be (automatically) extended to a wider class of functions. It is enough to have the integral (4.1) absolutely convergent or even convergent in a weaker sense. This however depends on the rate of growth of $\|G^\nu(u)\|$ at infinity. It turns out that it is related to the behavior of the semigroup near the imaginary axis $\Re z = 0$ and generally might be any subexponential function. If the growth is at most polynomial Theorem 4.1 can be essentially improved. This will be discussed in Section 6.

2. Let $\mathscr{A}$ be the (commutative) Banach algebra of operators on X, generated by the semigroup $\{a^z\}_{\Re z > 0}$. Suppose that $\mathscr{A}$ is semi-simple. Then the Gelfand space $M(\mathscr{A})$ can be identified with $\sigma(A)$ so that $\widehat{a^z}(t) = e^{-tz}$ and hence $\widehat{f(A)}(t) = f(t)$, $t \in \sigma(A)$, for any holomorphic function f which operates on A. The formula (4.1) and property (iii) of Theorem 4.1 implies then that $\Phi(f) \in \mathscr{A}$ for any smooth function f on $[0, \infty)$ with compact support and that $\widehat{\Phi(f)}(t) = f(t)$. This is also a consequence of (4.3) below.

3. If A is multiplication by the independent variable on $L^2(0, \infty)$, i.e. $Af(\lambda) = \lambda f(\lambda)$; or more generally, if A acts on a Hilbert space and has spectral decomposition $A = \int_0^\infty \lambda \, dE(\lambda)$, then $G^\nu(u)$ is multiplication by the function

$$\frac{1}{2\pi i} \int_{\Re z = 1} \frac{e^{-z\lambda}}{z^{\nu+1}} \, e^{uz} \, dz = \frac{1}{\Gamma(\nu+1)} \, (u - \lambda)_+^\nu, \tag{4.3}$$

and therefore $\Phi(f) = \int_0^\infty W^{\nu+1} f(u) \, G^\nu(u) \, du$ is multiplication by

$$\frac{1}{\Gamma(\nu+1)} \int_0^\infty W^{\nu+1} f(u) \, (u - \lambda)_+^\nu \, du = f(\lambda).$$

To check (4.3) use

$$\Gamma(\nu+1)\int_{\Re z=1}\frac{e^z}{z^{\nu+1}}\,dz=\int_0^\infty t^\nu e^{-t}\int_{\Re z=1}\frac{e^z}{z^{\nu+1}}\,dz\,dt$$

$$=\int_0^\infty e^{-t}\int_{\Re z=1/t}\frac{e^{tz}}{z^{\nu+1}}\,dz\,dt=\int_{\Re z=1/2}\frac{dz}{(1-z)z^{\nu+1}}$$

$$=-2\pi i\,\operatorname{res}\left(\frac{1}{(1-z)z^{\nu+1}},\,1\right)=2\pi i.$$

The above considerations are meant to illustrate the role played by the kernel G^ν. However, it will be made even more explicit after Proposition 4.4.

LEMMA 4.2. *The function $u\to\|G^\nu(u)\|$ is subexponential, i.e. for any $t>0$ there exists a constant C_t such that*

$$\|G^\nu(u)\|\leqslant C_t\,e^{tu}.$$

Proof. Since the function $z\to a^z z^{-\nu-1}e^{uz}$ is holomorphic in the strip $t\leqslant\Re z\leqslant 1$ and tends to zero at infinity we can change the path of integration in (4.2) to $\Re z=t$. ∎

Lemma 4.2 and an easy argument using both the Fubini and Residue theorems applied to (4.2) tells us that

$$a^z=z^{\nu+1}\int_0^\infty G^\nu(u)e^{-zu}\,du, \tag{4.4}$$

where the integral is absolutely convergent if $\Re z>0$. In other words, we have the following (cf. [HP, Theorem 10.6.2]):

A semigroup of operators on a Banach space which is holomorphic and of finite order α in every half plane $\Re z\geqslant t>0$ is for any $\nu>\alpha$ an absolutely convergent generalized Laplace transform (4.4) *of a norm-continuous operator valued function G^ν on $[0,\infty)$.*

COROLLARY 4.3. *In the notation of Theorem* 4.1 *assume that $1\leqslant p\leqslant 2$ and let $\varepsilon>0$. Then*

$$\|\Phi(f)\|\leqslant C_\varepsilon\left(\int_0^\infty |W^{\nu+1/p}f(u)\,e^{\varepsilon u}|^p\,du\right)^{1/p} \tag{4.5}$$

with a constant C_ε independent of f.

Proof. Put $\mu = \nu + 1/p$ and denote for simplicity by $\Phi_h(f)$, $h > 0$, the operator

$$\Phi_h(f) = \int_0^\infty \frac{W^\mu f(u) - W^\mu f(u+h)}{h} G^\mu(u)\, du.$$

Since

$$\lim_{h \to 0^+} \frac{W^\mu f(u) - W^\mu f(u+h)}{h} = -\frac{d}{du} W^\mu f(u) = W^{\mu+1} f(u),$$

we have that $\lim_{h \to 0+} \Phi_h(f) = \Phi(f)$ in the norm of $B(X)$. The only thing we must do now is to get an estimate of the form

$$\|\Phi_h(f)\| \leqslant C_\varepsilon \left(\int_0^\infty |W^\mu f(u) e^{\varepsilon u}|^p\, du \right)^{1/p}. \tag{4.6}$$

To do this first observe that $G^\mu(u-h) = 0$ when $u < h$ (cf. the beginning of the proof of Theorem 4.1). So

$$\begin{aligned}
\Phi_h(f) &= \int_0^\infty W^\mu f(u) \frac{G^\mu(u) - G^\mu(u-h)}{h}\, du \\
&= \frac{1}{2\pi i} \int_{\Re z = \varepsilon} \frac{a^z}{z^\mu} \int_0^1 e^{-thz}\, dt \int_0^\infty W^\mu f(u) e^{uz}\, du\, dz \\
&= \frac{1}{2\pi} \int_{-\infty}^\infty \frac{a^{\varepsilon + is}}{(\varepsilon + is)^\mu} \int_0^1 e^{-th(\varepsilon + is)}\, dt\, \hat{g}(s)\, ds,
\end{aligned}$$

where we denote by g the function $g(u) = W^\mu f(u) e^{\varepsilon u}$ for $u \geqslant 0$ and $g(u) = 0$ otherwise. Choose β such that $\alpha < \beta < \nu$. Then $\|a^{\varepsilon + is}\| \leqslant C|\varepsilon + is|^\beta$, and by the Hölder inequality and the Hausdorff–Young inequality applied to g we finally obtain

$$\begin{aligned}
\|\Phi_h(f)\| &\leqslant C \int_{-\infty}^\infty |\varepsilon + is|^{\beta - \mu}\, |\hat{g}(s)|\, ds \\
&\leqslant C \left(\int_{-\infty}^\infty |\varepsilon + is|^{p(\beta - \nu) - 1}\, ds \right)^{1/p} \|\hat{g}\|_{p'} \\
&\leqslant C_\varepsilon \|g\|_p. \quad \blacksquare
\end{aligned}$$

Next we are going to widen the class of functions on which Φ can operate indeed.

Definition. Let X be a Banach space and $\{a^z\}_{\Re z>0}$ a holomorphic (C_0) semigroup in $B(X)$ which satisfies (G_a). Let $-A$ denote the infinitesimal generator of the semigroup and Φ the operation defined by (4.1). We say that a function f in $L^\infty([0, \infty))$ *operates on A into $B(X)$* if it can be approximated in a suitable norm or topology by a sequence $\{f_n\}$ of smooth functions with compact supports so that such a convergence $f_n \to f$ implies that the sequence $\{\Phi(f_n)\}$ is convergent in the operator norm topology. The corresponding limit operator will be denoted then by $f(A)$. Since, on the set of all operating functions, the topology induced from the norm of $B(X)$ via the map $f \to f(A)$ generally may not separate points, we will always indicate the way of approximation.

Proposition 4.4. *In the notation of Theorem* 4.1 *assume that a measurable function $g\colon [0, \infty) \to \mathbf{C}$ for some $\varepsilon > 0$ satisfies*

$$\int_0^\infty |g(u)e^{\varepsilon u}|\, du < \infty.$$

Let f denote the Weyl fractional integral of g of order $v+1$, with $v > \alpha \geqslant 0$,

$$f(x) = \frac{1}{\Gamma(v+1)} \int_x^\infty (t-x)^v\, g(t)\, dt.$$

Then f operates on A and $f(A)$ is given by the absolutely convergent Bochner integral

$$f(A) = \int_0^\infty g(u)\, G^v(u)\, du.$$

Moreover

$$\|f(A)\| \leqslant C_\varepsilon \int_0^\infty |\, g(u)e^{\varepsilon u}|\, du, \tag{4.7}$$

with a constant C_ε independent of f.

Proof. Note that if $g \in C_c^\infty([0, \infty))$ then also $f \in C_c^\infty([0, \infty))$ and $g = W^{v+1}f$. Therefore $f(A) = \Phi(f)$. Let $\{g_n\}$ be a sequence in $C_c^\infty([0, \infty))$ which tends to g in the norm defined by the right hand side of (4.7), and let f_n be the Weyl integral of g_n of order $v+1$. Then it follows from Corollary 4.3 that $\{\Phi(f_n)\}$ tends to $\int_0^\infty g(u)\, G^v(u)\, du$ in the norm of $B(X)$. ∎

Remark. Suppose that $\theta > \nu$ and let R_λ^θ denote the Bochner–Riesz function defined by (3.2). Then $W^{\nu+1}R_\lambda^\theta = R_\lambda^{\theta-\nu-1}$ and

$$\int_0^\infty |R_\lambda^{\theta-\nu-1}(u)|\, e^u\, du = \frac{e^\lambda}{\Gamma(\theta-\nu)} \int_0^\lambda v^{\theta-\nu-1} e^{-v}\, dv \leqslant e^\lambda < \infty.$$

It follows from Proposition 4.4 that $R_\lambda^\theta(A)$ exists and

$$\begin{aligned} R_\lambda^\theta(A) &= \frac{1}{\Gamma(\theta-\nu)} \int_0^\lambda (\lambda-u)^{\theta-\nu-1}\, G^\nu(u)\, du \\ &= \frac{1}{2\pi i}\, \frac{1}{\Gamma(\theta-\nu)} \int_{\Re z=1} \int_0^\infty v^{\theta-\nu-1} e^{-vz}\, dv\, \frac{a^z}{z^{\nu+1}}\, e^{\lambda z}\, dz \\ &\quad - \frac{1}{2\pi i}\, \frac{1}{\Gamma(\theta-\nu)} \int_\lambda^\infty \int_{\Re z=1} \frac{a^z}{z^{\nu+1}}\, e^{(\lambda-v)z}\, dz\; v^{\theta-\nu-1}\, dv \\ &= G^\theta(\lambda), \end{aligned}$$

since the second integral is null by the Residues theorem.

Because of this, the operator-valued function $G^\nu(u)$ will be called here the *Bochner–Riesz kernel* associated to the semigroup $\{a^z\}_{\Re z>0}$, or alternatively, to the infinitesimal generator A.

Note that, when A is a well-bounded operator, there exists a decomposition of the identity $E(t)$, $t \geqslant 0$, for A in $B(X^*)$ such that

$$x^*(a^z x) = z \int_0^\infty e^{-zt} (E(t)x^*)(x)\, dt$$

for every $x \in X$, $x^* \in X^*$ (c.f. [deL2]). Since $-A$ generates a holomorphic (C_0) semigroup satisfying (G_1) (see ibid.), formulas (4.2) and (4.3) yield

$$\begin{aligned} x^*(G^\nu(\lambda)x) &= \int_0^\infty \frac{1}{2\pi i} \int_{\Re z=1} \frac{e^{(\lambda-t)z}}{z^\nu}\, dz (E(t)x^*)(x)\, dt \\ &= \frac{1}{\Gamma(\nu)} \int_0^\lambda (\lambda-t)^{\nu-1}\, (E(t)x^*)(x)\, dt \end{aligned}$$

for $\nu > 1$, $\lambda > 0$, and every $x \in X$, $x^* \in X^*$. This means that the decomposition of the identity E can be obtained as the Riemann–Liouville fractional derivative of order ν of the Bochner Riesz kernel G^ν.

In a similar way as in Proposition 4.4 we prove the following:

PROPOSITION 4.5. *In the notation of Theorem 4.1 assume that a measurable function* $g\colon [0,\infty)\to \mathbf{C}$ *for some* $\varepsilon>0$ *and some* $1\leqslant p\leqslant 2$ *satisfies*

$$\int_0^\infty |g(u)e^{\varepsilon u}|^p\,du<\infty.$$

Let f *denote the Weyl fractional integral of* g *of order* $\nu+1/p$, *where* $\nu>\alpha\geqslant 0$,

$$f(x)=\frac{1}{\Gamma(\nu+1/p)}\int_x^\infty (t-x)^{\nu-1+1/p}\,g(t)\,dt.$$

Then f *operates on* A *and* $f(A)$ *is given as the limit*

$$f(A)=\lim_{h\to 0^+}\int_0^\infty \frac{g(u)-g(u+h)}{h}\,G^{\nu+1/p}(u)\,du$$

in the operator norm of $B(X)$. *Moreover*

$$\|f(A)\|\leqslant C_\varepsilon\left(\int_0^\infty |g(u)\,e^{\varepsilon u}|^p\,du\right)^{1/p}, \tag{4.8}$$

with a constant C_ε *independent of* p *and of* f.

Proof. Let $\{g_n\}$ be a sequence in $C_c^\infty([0,\infty))$ which tends to g in the norm defined by the right hand side of (4.8), and let f_n be the corresponding Weyl integral of g_n of order $\nu+1/p$ for each n. It follows from (4.5) and (4.6) that the sequences $\Phi(f_n)$ and $\Phi_h(f_n)$, $h>0$, are convergent in $B(X)$, uniformly with respect to h. Since for each n $\Phi_h(f_n)\to\Phi(f_n)$ as $h\to 0^+$, the iterated limit $f(A)=\lim_{h\to 0^+}\lim_{n\to\infty}\Phi_h(f_n)$ exists and equals $\lim_{n\to\infty}\Phi(f_n)$. ∎

5. PROOF OF THEOREM 4.1

The main tool in the proof of Theorem 4.1 is the following:

PROPOSITION 5.1. *For any* $\tau>0$ *and any integer* $m>\nu$ *there exists a constant* C_τ *such that if* f *is a smooth function on* $[0,\infty)$ *with compact support and* f_0 *a holomorphic function in a neighborhood of* $[0,\infty)\cup\{\infty\}$ *with* $f_0(\infty)=0$ *then*

$$\|\Phi(f)a^\tau-f_0(A)a^\tau\|\leqslant C_\tau\,\|\tilde f-\tilde f_0\|_{C^{m+1}[0,1]},$$

where $\tilde f(x)=f((1/x)-1)$ *for* $0<x\leqslant 1$ *and* $\tilde f(0)=0$.

Proof. First observe that $G^\nu(u)=0$ whenever $u\leqslant 0$. It can be easily seen integrating (4.2) along $z=Re^{it}$, $|t|\leqslant \arctan R$, for $R>0$, instead of $z=1+it$, with $|t|\leqslant R$, and then passing with R to infinity. Therefore

$$G^\nu(u)=\frac{1}{\pi i}\int_{\Re z=1}\frac{a^z}{z^{\nu+1}}\sinh(uz)\,dz, \qquad u\geqslant 0.$$

For $\varepsilon>0$ define a new kernel G^ν_ε by

$$G^\nu_\varepsilon(u)=\frac{1}{\pi i}\int_{\Re z=1}(\varepsilon+z)^{-\nu}\,a^z\,\frac{\sinh(uz)}{z}\,dz.$$

Clearly all the $G^\nu_\varepsilon(u)$'s are bounded operators on X. Since the integral (4.2) is absolutely convergent, $|(\varepsilon+z)^{-\nu}|<|z^{-\nu}|$ and $(\varepsilon+z)^{-\nu}$ tends (uniformly) to $z^{-\nu}$ on $\{\Re z=1\}$ it follows by the dominated convergence theorem for vector-valued functions [HP, p. 83] that $\lim_{\varepsilon\to 0}G^\nu_\varepsilon(u)=G^\nu(u)$ for every $u\in\mathbf{R}$.

Fix $\tau>0$ and consider the operator $G^\nu_\varepsilon(u)a^\tau$. We claim that $\|G^\nu_\varepsilon(u)a^\tau\|=o(u^\delta)$ as $u\to\infty$ for any $\delta>0$. Indeed, the function $z\to(\varepsilon+z)^{-\nu}a^{\tau+z}(\sinh(uz)/z)$ is holomorphic in the strip $0\leqslant\Re z\leqslant\tau$ and tends to zero at infinity so we can write

$$\begin{aligned}G^\nu_\varepsilon(u)a^\tau&=\frac{1}{\pi i}\int_{\Re z=0}(\varepsilon+z)^{-\nu}\,a^{\tau+z}\,\frac{\sinh(uz)}{z}\,dz\\&=\frac{1}{\pi}\int_{-\infty}^{+\infty}(\varepsilon+is)^{-\nu}\,a^{\tau+is}\,\frac{\sin(us)}{s}\,ds.\end{aligned}$$

If $0<\delta<\nu-\alpha$ then, by (G_α),

$$\|G^\nu_\varepsilon(u)a^\tau\|\leqslant C\int_{-\infty}^{+\infty}\left|(\varepsilon+is)^{-\delta}\,\frac{\sin(us)}{s}\right|ds=O(u^\delta). \tag{5.1}$$

The Poisson integral formula applied to the bounded holomorphic function $z\to(\varepsilon+z)^{-\nu}\,a^{\tau+z}$ on the half-plane $\{\Re z>0\}$ yields

$$\begin{aligned}(\varepsilon+t)^{-\nu}\,a^{\tau+t}&=\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{t}{t^2+s^2}\,(\varepsilon+is)^{-\nu}\,a^{\tau+is}\,ds\\&=\frac{1}{\pi}\int_{-\infty}^{+\infty}\left(\int_0^{+\infty}t\,\frac{\sin(us)}{s}\,e^{-tu}\,du\right)(\varepsilon+is)^{-\nu}\,a^{\tau+is}\,ds\\&=\int_0^{+\infty}te^{-tu}G^\nu_\varepsilon(u)a^\tau\,du\end{aligned}$$

for any $t>0$. Hence

$$a^{\tau+t}=\int_0^{+\infty} t(\varepsilon+t)^{\nu}\, e^{-tu} G_\varepsilon^\nu(u)\, a^\tau\, du. \tag{5.2}$$

For any integer $n \geqslant 1$ the operator $(1+A)^{-n}$ can be expressed as a Laplace integral

$$(1+A)^{-n}=\frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1} e^{-t} a^t\, dt\, .$$

Thus by (5.2) and Lemma 5.2 below we get

$$\begin{aligned}
(1+A)^{-n}\, a^\tau &= \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1} e^{-t} a^{\tau+t}\, dt \\
&= \int_0^{+\infty} \frac{1}{(n-1)!}\int_0^{+\infty} t^n(\varepsilon+t)^\nu\, e^{-t(1+u)}\, dt\, G_\varepsilon^\nu(u)\, a^\tau\, du \\
&= \int_0^{+\infty} e^{\varepsilon u}\, W^\nu(n(1+u)^{-n-1}\, e^{-\varepsilon u})\, G_\varepsilon^\nu(u)\, a^\tau\, du \\
&= -\int_0^{+\infty} e^{\varepsilon u}\, W^\nu\left(e^{-\varepsilon u}\frac{d}{du}\,(1+u)^{-n}\right) G_\varepsilon^\nu(u)\, a^\tau\, du.
\end{aligned}$$

Observe that the last integral is absolutely convergent since $\|G_\varepsilon^\nu(u)\, a^\tau\| = o(u^\delta)$ for some $0<\delta<1$, as we have already seen in (5.1), and the first factor in the integral is $O(u^{-n-1})$ with $n \geqslant 1$.

If p is a polynomial with $p(0)=0$ it follows from the last equality that

$$p((1+A)^{-1})\, a^\tau = -\int_0^{\infty} e^{\varepsilon u}\, W^\nu\left(e^{-\varepsilon u}\,\frac{d}{du}\, p((1+u)^{-1})\right) G_\varepsilon^\nu(u)\, a^\tau\, du$$

for arbitrary $\varepsilon \geqslant 0$. The right hand side of the last equality defines a bounded operator on X even if instead of p we put any smooth function h on $[0,1]$. Let us denote the so defined operator by $\Psi(h)$. We claim that for any integer $m \geqslant \nu$

$$\|\Psi(h)\| \leqslant C\, \|h\|_{C^{m+1}[0,\,1]}\, , \tag{5.3}$$

where C is a constant independent of h, although it may depend on ε, τ and m. Indeed, by (5.1) we have

$$\|\Psi(h)\| \leqslant C \int_0^{\infty} |W^\nu g(u)|\, u^\delta e^{\varepsilon u}\, du,$$

where

$$g(u)=e^{-\varepsilon u}\,\frac{d}{du}\,h\left(\frac{1}{1+u}\right).$$

When $v<m$ write $W^{v}=W^{v-m}W^{m}$. Then

$$\begin{aligned}\int_0^{\infty}|W^{v}g(u)|\,u^{\delta}e^{\varepsilon u}\,du &\leqslant \frac{1}{\Gamma(m-v)}\int_0^{\infty}\int_u^{\infty}(t-u)^{m-v-1}\,|W^{m}g(t)|\,u^{\delta}e^{\varepsilon u}\,dt\,du\\ &\leqslant \frac{1}{\Gamma(m-v)}\int_0^{\infty}\int_0^{t}(t-u)^{m-v-1}\,u^{\delta}\,du\,e^{\varepsilon t}\,|W^{m}g(t)|\,dt\\ &=C'\int_0^{\infty}|W^{m}g(t)|\,t^{m-v+\delta}e^{\varepsilon t}\,dt.\end{aligned}$$

But $W^{m}=(-1)^{m}\,(d^{m}/du^{m})$, thus

$$e^{\varepsilon u}W^{m}g(u)=\sum_{k=0}^{m}\binom{m}{k}\varepsilon^{m-k}(-1)^{k}\,\frac{d^{k+1}}{du^{k+1}}\,h\left(\frac{1}{1+u}\right)$$

and so

$$e^{\varepsilon u}|W^{m}g(u)|\leqslant\frac{1}{(1+u)^{2}}\sum_{k=0}^{m}\alpha_k\left|h^{(k+1)}\left(\frac{1}{1+u}\right)\right|$$

with coefficients $\alpha_0, \alpha_1, \alpha_2, ..., \alpha_m$ independent of h. This implies (5.3) if m and Y are chosen so that $m-v+\delta<1$.

Now suppose that the functions f and f_0 satisfies the assumptions of the proposition. Passing from f_0 to $\tilde{f}_0$ we see that there exists a compact set K in $\mathbf{C}$, containing $[0, 1]$ in its interior and such that $\tilde{f}_0$ has a holomorphic extension to K (which will be also denoted by $\tilde{f}_0$). Choose a sequence $p_1, p_2, ...$ of polynomials (without constant terms) which converges to $\tilde{f}_0$ uniformly on K. Then $p_n\to\tilde{f}_0$ in $C^{m+1}[0, 1]$ and also $p_n((1+A)^{-1})\to\tilde{f}_0((1+A)^{-1})=f_0(A)$ in the operator norm. By (5.3) this implies that $\Psi(\tilde{f}_0)=f_0(A)a^{\tau}$.

Assume now that $p_n\to\tilde{f}$ in $C^{m+1}[0, 1]$ and let $\Phi_{\varepsilon}(f)$ denote the operator

$$\Phi_{\varepsilon(}f)=-\int_0^{\infty}e^{\varepsilon u}W^{v}(e^{-\varepsilon u}f'(u))\,G_{\varepsilon}^{v}(u)\,du.$$

By (5.3) $p_n((1+A)^{-1})a^{\tau}$ converges in the operator norm to $\Phi_{\varepsilon}(f)a^{\tau}$ when $n\to\infty$. But this implies that $\Phi_{\varepsilon}(f)$ is independent of ε, and so by

Lemma 5.3 below it must be $\Phi_\varepsilon(f)=\Phi(f)$. Consequently $\Psi(\tilde{f})=\Phi(f)a^\tau$, which gives

$$\|\Phi(f)a^\tau - f_0(A)a^\tau\| = \|\Psi(\tilde{f}-\tilde{f}_0)\| \leqslant C\,\|\tilde{f}-\tilde{f}_0\|_{C^{m+1}[0,1]}. \quad \blacksquare$$

LEMMA 5.2. *We have*

$$e^{\varepsilon x}W^\nu((1+x)^{-\delta}\,e^{-\varepsilon x}) = \frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}(\varepsilon+t)^\nu\,e^{-t(1+x)}\,dt = O(x^{-\delta})$$

as $x\to\infty$, *for any positive* ε, ν, δ.

Proof. Let

$$\varphi(x) = \frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}(\varepsilon+t)^\nu\,e^{-t(1+x)}\,dt.$$

Then

$$\begin{aligned}
&e^{\varepsilon x}W^{-\nu}(e^{-\varepsilon x}\varphi(x))\\
&\quad=\frac{e^{\varepsilon x}}{\Gamma(\nu)}\int_0^\infty s^{\nu-1}e^{-\varepsilon(s+x)}\varphi(s+x)\,ds\\
&\quad=\frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}(\varepsilon+t)^\nu\,e^{-t(1+x)}\,\frac{1}{\Gamma(\nu)}\int_0^\infty s^{\nu-1}e^{-s(\varepsilon+t)}\,ds\,dt\\
&\quad=\frac{1}{\Gamma(\delta)}\int_0^\infty t^{\delta-1}e^{-t(1+x)}\,dt\\
&\quad=\frac{1}{(1+x)^\delta}. \quad \blacksquare
\end{aligned}$$

LEMMA 5.3. *Let* ε, $\nu>0$ *and let* $f\in\mathscr{E}$. *Then*

$$e^{\varepsilon x}W^\nu(e^{-\varepsilon x}f(x)) = \sum_{k=0}^\infty \binom{\nu}{k}\varepsilon^k\,W^{\nu-k}f(x)$$

and tends to $W^\nu f(x)$ *uniformly when* $\varepsilon\to 0$.

Proof. Denote $g(x)=e^{-\varepsilon x}W^\nu f(x)$. Then

$$e^{-\varepsilon x}f(x) = \frac{1}{\Gamma(\nu)}\int_0^\infty t^{\nu-1}e^{\varepsilon t}g(t+x)\,dt = \sum_{k=0}^\infty \frac{\Gamma(\nu+k)}{\Gamma(\nu)k!}\,\varepsilon^k\,W^{-\nu-k}g(x).$$

Since the series is convergent in the topology of $\mathscr{E}$, we get

$$
\begin{aligned}
e^{\varepsilon x} W^{\nu}(e^{-\varepsilon x} f(x)) &= \sum_{k=0}^{\infty} \frac{\Gamma(\nu+k)}{\Gamma(\nu)\, k!}\, \varepsilon^k e^{\varepsilon x} W^{-k} g(x) \\
&= W^{\nu} f(x) + \varepsilon \int_0^{\infty} \varphi_{\nu}(\varepsilon t) W^{\nu} f(t+x)\, dt
\end{aligned}
$$

with

$$
\varphi_{\nu}(t) = \sum_{k=0}^{\infty} \binom{k+\nu}{k+1} \frac{t^k}{k!}\, e^{-t}.
$$

The uniform convergence follows since $\varphi_{\nu}(t)$ is bounded when $0<\nu \leqslant 1$ and is $O(t^{\nu-1})$ when $\nu>1$. Indeed, the formula

$$
\frac{d}{dt}\, [t^{1-\nu} \varphi_{\nu}(t)] = -\nu\, t^{-\nu} \varphi_{\nu-1}(t)
$$

implies that $t^{1-\nu}\varphi_{\nu}(t)$ is a positive decreasing function. Now observe (Kummer's transformation) that

$$
\varphi_{\nu}(t) = \sum_{k=0}^{\infty} \binom{\nu}{k+1} \frac{t^k}{k!},
$$

which gives the final formula immediately. ∎

Proof of Theorem 4.1. Let f be a smooth function on $[0, \infty)$ with compact support and z a complex number, $\Re z>0$. Set $\tilde{f}(t)=f((1/t)-1)$ for $0<t \leqslant 1$ and $\tilde{f}(0)=0$. Choose an integer $m \geqslant \nu$ and let $p_1, p_2, \ldots$ be a sequence of polynomials without constant terms, which converges to $\tilde{f}$ uniformly on $[0,1]$ together with all their derivatives up to order $m+1$. Define a sequence of holomorphic functions by $f_n(\lambda)=p_n(1/(1+\lambda))$, $\lambda \neq -1$. Finally choose τ such that $0<\tau<\Re z$. Proposition 5.1 implies that $f_n(A) a^{\tau} = p_n((1+A)^{-1}) a^{\tau} \rightarrow \Phi(f) a^{\tau}$ in the operator norm. Since $a^z = a^{\tau} a^{z-\tau}$, the same is true for $f_n(A) a^z$. This proves (iii).

To prove (ii) observe that

$$
\Phi(f)\Phi(g) a^2 = \lim_{n \rightarrow \infty} f_n(A) a^1 \cdot \lim_{n \rightarrow \infty} g_n(A) a^1 = \lim_{n \rightarrow \infty} (f_n g_n)(A) a^1 = \Phi(fg) a^2
$$

with a proper choice of holomorphic functions f_n, g_n. It gives $\Phi(f)\Phi(g)=\Phi(fg)$ since they coincide on a dense in X subspace $X_2=\{a^2 x \colon x \in X\}$. The second part of (ii) follows in a similar way, for if

$h(u)=f(u)+g(u)=(1+u)f(u)$, $u\geqslant 0$, and $p_1, p_2, \ldots$ is a sequence of polynomials such that $p_n \to \tilde{h}$ in $C^{m+1}[0,1]$ for some $m \geqslant \nu$ then $q_n \to \tilde{f}$ in $C^{m+1}[0,1]$, where $q_n(t)=t\,p_n(t)$, and both sequences $p_n((1+A)^{-1})a^1$, $q_n((1+A)^{-1})a^1$ converge in $B(X)$ to $\Phi(h)a^1$ and $\Phi(f)a^1$ respectively. But $q_n((1+A)^{-1})=(1+A)^{-1}p_n((1+A)^{-1})=p_n((1+A)^{-1})(1+A)^{-1}$, thus

$$(1+A)^{-1}\Phi(f+g)=\Phi(f+g)(1+A)^{-1}=\Phi(f).$$

This implies that $\Phi(f)(X)\subset \mathscr{D}(A)$ and $\Phi(f)A\subset A\Phi(f)=\Phi(g)$.

Point (i) is an immediate consequence of (iii), because $f_n(A)a^z \to \Phi(f)a^z$ independently if $\Phi(f)$ in (4.1) is defined with exponent μ or ν. ∎

6. THE CASE OF SUBHOMOGENEOUS SEMIGROUPS

In many cases a semigroup $\{a^z\}_{\Re z>0}$ is *homogeneous*, which implies that

$$\|a^{tz}\|=\|a^z\| \qquad \text{for every} \quad t>0 \text{ and } \Re z>0.$$

Under this assumption, for a real number α, property (G_α) takes a stronger form

$$\|a^z\|\leqslant C_\nu(|z|/\Re z)^\nu \qquad \text{for} \quad \Re z>0 \text{ and every } \nu>\alpha. \tag{HG_α}$$

However there are also non-homogeneous semigroups satisfying (HG_α). We will call them *subhomogeneous* with growth exponent α. For subhomogeneous semigroups the estimates obtained in Section 5 can be improved.

LEMMA 6.1. *Let $\{a^z\}_{\Re z>0}$ be a subhomogeneous semigroup with growth exponent α and let $\nu>\alpha$. Then the Bochner–Riesz kernel G^ν associated to $\{a^z\}_{\Re z>0}$ satisfies*

$$\sup_{u>0}\|G^\nu(u)\|/u^\nu<\infty.$$

Proof. Take μ such that $\alpha<\mu<\nu$. Changing the path of integration in (4.2) to $\Re z=1/u$ we get

$$\begin{aligned}\|G^\nu(u)\| &\leqslant u^\nu\,\frac{e}{2\pi}\int_{-\infty}^{+\infty}\|a^{(1+it)/u}\|\,|1+it|^{-\nu-1}\,dt\\ &\leqslant u^\nu C_\mu\,\frac{e}{2\pi}\int_{-\infty}^{+\infty}|1+it|^{\mu-\nu-1}\,dt. \quad ∎\end{aligned}$$

As an immediate consequence of Theorem 4.1 and Lemma 6.1 we obtain that if $\nu > \alpha$ then any $AC^{(\nu+1)}$ function operates on A to $B(X)$. In fact, because of Corollary 3.3, we can state more.

THEOREM 6.2. *Let $-A$ be the infinitesimal generator of a holomorphic (C_0) semigroup $\{a^z\}_{\Re z>0}$ of operators on a Banach space X which satisfies (HG_α). Suppose that f is a complex function on $[0, \infty)$ which vanishes at infinity and has a holomorphic extension to a neighborhood of $[0, \infty) \cup \{\infty\}$. Let $f(A)$ be the bounded operator on X related to f by the holomorphic calculus formula* (2.1). *Then for every $\nu > \alpha$ we have*

$$f(A) = \int_0^\infty W^{\nu+1} f(u)\, G^\nu(u)\, du, \tag{6.1}$$

where G^ν is the Bochner–Riesz kernel defined by (4.2). *The integral is absolutely convergent and*

$$\|f(A)\| \leqslant C \int_0^\infty \left| W^{\nu+1} f(x) \right| x^\nu\, dx \tag{6.2}$$

with a constant C independent of f. ∎

Remarks. 1. Observe that Lemma 6.1 and Theorem 6.2 become true if instead of (HG_α) we only assume that

$$\sup_{\varepsilon>0} \int_{-\infty}^{+\infty} \|a^{\varepsilon(1+it)}\|\, |1+it|^{-\nu-1}\, dt < \infty$$

for every $\nu > \alpha$.

2. Theorem 6.2 is "almost" invertible. If $-A$ is the infinitesimal generator of a semigroup $\{a^t\}_{t>0}$ in $B(X)$, $\sigma(A) \subset [0, \infty)$ and $\|f(A)\| \leqslant C \|f\|_{(\nu+1)}$ for holomorphic functions then the semigroup can be extended to the half-plane $\Re z > 0$ with $\|a^z\| \leqslant C(|z|/\Re z)^{\nu+1}$ (see also Section 7.B).

Theorem 6.2 is more powerful than it might look at first glance. Formula (6.1) allows us to go out of the class of holomorphic functions. For any $\nu > \alpha$ the map $f \to f(A)$ extends uniquely to a bounded Banach algebra homomorphism of $AC^{(\nu+1)}$ into $B(X)$ such that $f(A) A \subset A f(A) = g(A)$ whenever $f, g \in AC^{(\nu+1)}$ and $g(u) = u \cdot f(u)$. In the notation of section 7.B it means that the operator A has a proper $AC^{(\nu+1)}$ functional calculus. The additional advantage of Theorem 6.2 is that the calculus formula is given explicitly (formula (6.1)). Also very often in practice, for particular operators A, the Bochner–Riesz kernel $G^\nu(u)$ can be given in a form of closed analytic expression. Of course instead of fractional derivatives of functions one can use ordinary derivatives of high enough order.

If one wants to save on the degree of smoothness of a function f to define an operator $f(A)$ we propose another, stronger but less explicit functional calculus for A related to the algebra $AC_{2,1}^{(\nu+1/2)}$. The following theorem is the analog of Proposition 4.5 in the subhomogeneous case.

THEOREM 6.3. *Suppose that the semigroup in Theorem* 4.1 *satisfies* (HG_α), *and take* $\nu > \alpha$. *If f is a smooth function with compact support then*

$$\|f(A)\| \leqslant C \int_0^\infty \left[\int_y^{2y} |W^{\nu+1/2} f(x) x^\nu|^2 \, dx \right]^{1/2} \frac{dy}{y}. \tag{6.3}$$

Thus the correspondence $f \to f(A)$ extends to a bounded, multiplicative, linear mapping from $AC_{2,1}^{(\nu+1/2)}$ into $B(X)$.

Proof. First observe that the estimate (4.5) in Corollary 4.3 can be improved to the following

$$\|f(A)\| \leqslant C \inf_{t>0} t^{-\nu} \left(\int_0^\infty |W^{\nu+1/2} f(x) e^{tx}|^2 \, dx \right)^{1/2}. \tag{6.4}$$

To see this, just repeat the proof of the corollary, but expressing $G^\nu(u)$ as

$$G^\nu(u) = \frac{1}{2\pi i} \int_{\Re z = t} \frac{a^z}{z^{\nu+1}} e^{uz} \, dz,$$

with $t > 0$, making the change of variable $z = t\lambda$, and applying the subhomogeneity of $\{a^z\}_{\Re z > 0}$.

Now take a nonnegative C^∞ function φ with support in $(\frac{1}{2}, 2)$ and such that $\sum_{-\infty}^{\infty} \varphi_k(x) = 1$ for $x > 0$, where $\varphi_k(x) = \varphi(2^k x)$. For any integer k put

$$f_k = W^{-\nu-1/2}(\varphi_k W^{\nu+1/2} f).$$

Then each f_k lies in $C_c^\infty([0, \infty))$, $f = \sum_{-\infty}^{\infty} f_k$, and by (6.4)

$$\|f_k(A)\| \leqslant C \inf_{t>0} t^{-\nu} \exp(t\, 2^{k+1}) \left(\int_{2^{k-1}}^{2^{k+1}} |W^{\nu+1/2} f(x)|^2 dx \right)^{1/2}.$$

Thus

$$\|f(A)\| \leqslant \sum_{-\infty}^{\infty} \|f_k(A)\| \leqslant C' \sum_{-\infty}^{\infty} 2^{k\nu} \left(\int_{2^{k-1}}^{2^{k+1}} |W^{\nu+1/2} f(x)|^2 \, dx \right)^{1/2}$$
$$\leqslant C'' \|f\|_{(\nu+1/2),2,1}.$$

The remainder of the proof follows from the density of $C_c^\infty([0,\infty))$ in $AC_{2,1}^{(\nu+1/2)}$. ▮

From (3.4) we get immediately the following corollary.

COROLLARY 6.4. *Under the assumptions of Theorem* 6.3,

$$\|f(A)\| \leqslant C\left(\int_0^\infty |W^{\nu+1/2}f(x)\, x^\nu(1+|\log x|)|^2\, dx\right)^{1/2},$$

where C is a constant independent of f.

7. APPLICATIONS AND EXAMPLES

We present here some applications of the functional calculus that we have established in Sections 4 and 6. Then we discuss the classical semigroups on $\mathbf{R}^n$ and on stratified Lie groups. These examples can be used to test the strength of the method. They gave us also the initial motivations for the considerations.

A. *Fractional Powers of Generators*

The calculus developed in Section 6 allows us to introduce positive fractional powers of generators of semigroups in a quite simple way.

LEMMA 7.1. *Fix $\theta>0$. The family $\{e_{z,\theta}\}_{\Re z>0}$, where $e_{z,\theta}(u)=\exp(-zu^\theta)$, $u\geqslant 0$, forms a homogeneous holomorphic (C_0) semigroup of functions in every algebra $AC^{(\nu)}$, $\nu>0$, satisfying*

$$\|e_{z,\theta}\|_{(\nu)} \leqslant C_{\nu,\theta}(|z|/\Re z)^\nu.$$

Proof. Clearly $e_{z,\theta}\in AC^{(\nu)}$ and by Proposition 3.5

$$\|e_{z,\theta}\|_{(\nu)} \leqslant C_{\nu,\theta}\|e_{z,1}\|_{(\nu)} = C_{\nu,\theta}(|z|/\Re z)^\nu.$$

Also, it is easy to check that $\{e_{z,\theta}\}_{\Re z>0}$ is holomorphic, so it remains to show that condition (C_0) is satisfied.

Let f be a smooth function with compact support. Lemma 5.3 implies that $W^\nu(e_{\varepsilon,1}\cdot f)$ tends to $W^\nu f$ uniformly on $(0,\infty)$, but both functions are supported by a fixed compact set, thus

$$\lim_{\varepsilon\to 0}\|e_{\varepsilon,1}\cdot f - f\|_{(\nu)}=0.$$

The convergence holds then on all of $AC^{(\nu)}$, for smooth compactly supported functions are dense there. Finally $\|e_{\varepsilon,\theta}\cdot f-f\|_{(\nu)}\leqslant C_{\nu,\theta}\|e_{\varepsilon,1}\cdot \tilde{f}-\tilde{f}\|_{(\nu)}$, where $f(u)=f(u^{1/\theta})$, by Proposition 3.5. ∎

DEFINITION. Assume that $-A$ generates a holomorphic (C_0) semigroup of operators, on a Banach space X, which satisfies (HG_α) and fix $\theta>0$. Theorem 6.2 and Lemma 7.1 with $\nu>\alpha$ imply that $\{e_{z,\theta}(A)\}_{\Re z>0}$ is a holomorphic (C_0) semigroup in $B(X)$, subhomogeneous with growth exponent $\alpha_\theta\leqslant\alpha+1$ (in fact from Theorem 6.3 and Proposition 3.9 it follows that $\alpha_\theta\leqslant\alpha+1/2$). Define $-A^\theta$ to be the infinitesimal generator of this semigroup. The $AC^{(\nu)}$ (e.g. when $\nu>\alpha+2$) functional calculus for $-A^\theta$ is like that for $-A$ with the variable u^θ. In particular $e_{z,\theta_2}(A^{\theta_1})=e_{z,\theta_1\theta_2}(A)$, so $(A^{\theta_1})^{\theta_2}=A^{\theta_1\theta_2}$.

LEMMA 7.2. *If n is a natural number then A^n coincides with the usual nth power of A.*

Proof. To distinguish these two operators let us denote by B the usual n-th power of A. If λ is a complex number, $\lambda\notin[0,\infty)$, and $\lambda_1,\lambda_2,\ldots,\lambda_n$ are the n-th roots of λ then

$$\lambda-B=(-1)^{n-1}\prod_{k=1}^{n}(\lambda_k-A)$$

is invertible. Thus B is a closed operator with spectrum in $[0,\infty)$. Moreover, since $(\lambda-A)^{-1}=r_{\lambda,1}(A)$, where $r_{\lambda,\theta}$ is the function $r_{\lambda,\theta}(u)=1/(\lambda-u^\theta)$, we have

$$(\lambda-B)^{-1}=\left[(-1)^{n-1}\prod_{k=1}^{n}r_{\lambda_k,1}\right](A)=r_{\lambda,n}(A)=(\lambda-A^n)^{-1}.$$

It follows that $\mathscr{D}(B)=(\lambda-B)^{-1}(X)=(\lambda-A^n)^{-1}(X)=\mathscr{D}(A^n)$ and $Bx=A^nx$ for $x\in\mathscr{D}(B)$. ∎

For $0<\theta<1$ Balakrishnan (see [SKM, p. 120], [Y, p. 259] or [G, p. 62]) has defined the θ-th fractional power of A as $-B$, where B is the infinitesimal generator of the (C_0) semigroup $\{a_\theta^t\}_{t>0}$ given by

$$a_\theta^t x=\int_0^\infty g_{t,\theta}(u)\,a^u x\,du,\qquad x\in X,$$

and

$$g_{t,\theta}(u)=\frac{1}{2\pi i}\int_{\Re z=\sigma}\exp(uz-tz^\theta)\,dz,\qquad u\geqslant 0,\ \sigma>0.$$

PROPOSITION 7.3. *If* $0<\theta<1$ *then* A^θ *coincides with Balakrishnan's* θ*th fractional power of* A.

Proof. We will show that if $x \in \mathcal{D}(A)$ then $x \in \mathcal{D}(A^\theta)$ and

$$A^\theta x = \frac{1}{\Gamma(-\theta)} \int_0^\infty t^{-\theta-1}(a^t x - x)\,dt, \tag{7.1}$$

the integral being absolutely convergent. Exactly the same is valid for Balakrishnan's θth power of A. Since both operators are closed this will imply that they are equal.

Note that, since a^τ is injective for any $\tau>0$ ([deL3, Lemma 8.8]), it is enough to show (7.1) for vectors of the form $x=a^\tau y$, $\tau>0$, $y \in \mathcal{D}(A)$.

So let $x=a^\tau y$, where $\tau>0$ and $y \in X$. Then $x \in \mathcal{D}(A^\theta)$ and according to the $AC^{(\nu)}$-functional calculus of Theorem 6.2

$$A^\theta x = A^\theta e_\tau(A)\, y = \int_0^\infty W^{\nu+1}(u^\theta e^{-\tau u})\, G^\nu(u)\, y\, du,$$

where $e_\tau(u)=e^{-\tau u}$, $u>0$. By Marchaud's formula (Proposition 2.1) we have

$$\begin{aligned} u^\theta e^{-\tau u} = W^\theta e_u(\tau) &= \frac{-1}{\Gamma(-\theta)} \int_0^\infty t^{-\theta-1}\, \Delta_t e_u(\tau)\, dt \\ &= \frac{1}{\Gamma(-\theta)} \int_0^\infty t^{-\theta-1}(e_{t+\tau}(u) - e_\tau(u))\, dt, \end{aligned}$$

thus

$$\begin{aligned} A^\theta x &= \frac{1}{\Gamma(-\theta)} \int_0^\infty t^{-\theta-1} \int_0^\infty W^{\nu+1}(e_{t+\tau} - e_\tau)(u)\, G^\nu(u)\, y\, du\, dt \\ &= \frac{1}{\Gamma(-\theta)} \int_0^\infty t^{-\theta-1}(a^t x - x)\, dt. \quad \blacksquare \end{aligned}$$

B. *$AC^{(\nu)}$ Calculus*

Suppose that A is a densely defined, closed operator on a Banach space X and let $\nu \geqslant 1$. We shall say that A *has an* $AC^{(\nu)}$ *calculus* if there exists a bounded Banach algebra homomorphism $\Phi: AC^{(\nu)} \to B(X)$ with $\Phi(f)A \subset A\Phi(f) = \Phi(g)$ whenever $f, g \in AC^{(\nu)}$ and $g(t) = t \cdot f(t)$. If $\Phi(AC^{(\nu)})X$ is dense in X, the calculus is called *proper*.

LEMMA 7.4. *A densely defined, closed operator A on a Banach space has at most one proper $AC^{(v)}$ calculus. In that case $\Phi(f)$ coincides with $f(A)$ (defined by (2.1)) for every function $f \in AC^{(v)}$ which has a holomorphic extension to a neighborhood of $[0, \infty) \cup \{\infty\}$.*

Proof. Fix in X an element x of the form $x = \Phi(h)y$, where h is a function in $AC^{(v)}$ with compact support and $y \in X$. Note that $x \in \mathscr{D}(A)$ and $Ax = \Phi(g)y$, where $g(t) = t \cdot h(t)$. For any complex number $z \notin [0, \infty)$ the function $r_z(t) = 1/(z-t)$ lies in $AC^{(v)}$ and $(z-A)\,\Phi(r_z)x = \Phi(r_z)(z-A)x = x$. Since the set of elements in X of the form $x = \Phi(h)y$ is dense in X, we conclude that $\Phi(r_z) = (z-A)^{-1}$. We get in particular $\sigma(A) \subset [0, \infty)$. If f is a function in $AC^{(v)}$ which has a holomorphic extension to the closure of a neighborhood Ω of $[0, \infty) \cup \{\infty\}$, then

$$f(t) = \int_{\partial\Omega} f(z)\, r_z(t)\, dz, \qquad t \geqslant 0,$$

and so

$$\Phi(f) = \int_{\partial\Omega} f(z)\, \Phi(r_z)\, dz = f(A)$$

by (2.1). The uniqueness of Φ follows from Corollary 3.3. ∎

The next result supplies a characterization of the existence of a proper $AC^{(v)}$ calculus Φ for A (cf. [deL2] for $v = 1$).

PROPOSITION 7.5. *Let A be a densily defined, closed operator on a Banach space and let $\alpha \geqslant 1$. Then A has a proper $AC^{(v)}$ calculus for every $v > \alpha$ if and only if $-A$ generates a holomorphic subhomogeneous (C_0) semigroup $\{a^z\}_{\Re z > 0}$ in $B(X)$ with growth exponent $\leqslant \alpha$ and such that for any $v > \alpha$ the kernel*

$$G^v(u) = \frac{1}{2\pi i} \int_{\Re z = 1} \frac{a^z}{z^{v+1}}\, e^{uz}\, dz$$

is continuously differentiable on $(0, \infty)$ with $u^{1-v} \|\,(d/du)\, G^v(u)\|$ bounded.

Proof. Suppose that A has a proper $AC^{(v)}$ calculus. Following the arguments of the preceding proof we see that for any complex number $z \neq 0$ with $|\arg z| < \pi$ the operator $z(z+A)^{-1}$ exists and $\|z(z+A)^{-1}\| = \|z\Phi(r_{-z})\| \leqslant C\xi(z)$, where

$$\xi(z) = \|zr_{-z}\|_{(v)} = v \int_0^\infty \frac{|z|\, t^{v-1}}{|z+t|^{v+1}}\, dt.$$

Since $\xi(z)$ is continuous in z and $\xi(\lambda z)=\xi(z)$ for any $\lambda>0$, we see that

$$\sup\{\|z(z+A)^{-1}\| : |\arg z|\leqslant\theta\}<\infty$$

whenever $0<\theta<\pi$. This implies (see [G, Theorem 5.4]) that $-A$ generates in $B(X)$ a holomorphic (C_0) semigroup $\{T(z)\}_{\Re z>0}$.

There is also in $B(X)$ the holomorphic semigroup $\{a^z\}_{\Re z>0}$ defined as $a^z=\Phi(e_z)$ and $e_z(u)=e^{-zu}$, $u\geqslant 0$. Clearly $\|a^z\|\leqslant C\|e_z\|_{(v)}=C(|z|/\Re z)^v$ and for $\Re\lambda>0$ we have

$$\int_0^\infty e^{-\lambda t}a^t\,dt=\Phi\left(\int_0^\infty e^{-\lambda t}e_t\,dt\right)=\Phi(-r_{-\lambda})$$

$$=(\lambda+A)^{-1}=\int_0^\infty e^{-\lambda t}\,T(t)\,dt.$$

Hence $a^z=T(z)$, $\Re z>0$, by the injectivity of the Laplace transform. If the $AC^{(v)}$ calculus holds for any $v>\alpha$, then $\{a^z\}_{\Re z>0}$ satisfies (C_0) and (HG_α).

Assume that $v>\alpha$ and choose μ, κ so that $v>\mu>\kappa>\alpha$. Note that the Bochner–Riesz functions $R_t^{\mu-1}$, $t\geqslant 0$, defined by (3.1) all are in $AC^{(\kappa)}$. Thus the operator kernel $\Phi(R_t^{\mu-1})$ is well defined, continuous in t and $\|\Phi(R_t^{\mu-1})\|\leqslant Ct^{\mu-1}$. Since $e_z=z^\mu\int_0^\infty e^{-tz}R_t^{\mu-1}\,dt$ in $AC^{(\kappa)}$ when $\Re z>0$, we have

$$\frac{a^z}{z^\mu}=\int_0^\infty e^{-tz}\Phi(R_t^{\mu-1})\,dt$$

and so

$$G^v(u)=\int_0^\infty\left(\frac{1}{2\pi i}\int_{\Re z=1}\frac{e^{(u-t)z}}{z^{v-\mu+1}}\,dz\right)\Phi(R_t^{\mu-1})\,dt$$

$$=\int_0^u (u-t)^{v-\mu}\,\Phi(R_t^{\mu-1})\,dt$$

by (4.3). It follows that there exists $(d/du)\,G^v(u)=(v-\mu)\int_0^u (u-t)^{v-\mu-1}$ $\Phi(R_t^{\mu-1})\,dt$ and it is a norm-continuous function in u. Furthermore,

$$\left\|\frac{d}{du}\,G^v(u)\right\|\leqslant C(v-\mu)\int_0^u (u-t)^{v-\mu-1}\,t^{\mu-1}\,dt=C_1u^{v-1}$$

for some constant C_1.

Conversely, suppose that $-A$ generates a semigroup $\{a^z\}_{\Re z>0}$ with all the properties. Put $\Phi(f)=\int_0^\infty W^vf(u)(d/du)\,G^v(u)\,du\in B(X)$ when $v>\alpha$

and $f \in AC^{(\nu)}$. If $f \in AC^{(\nu+1)}$ we have indeed that $\Phi(f) = \int_0^\infty W^{\nu+1} f(u)$ $G^\nu(u)\,du$, and Theorem 6.2 tells us that Φ defines a bounded homomorphism from $AC^{(\nu+1)}$ into $B(X)$ such that $\Phi(f)A \subset A\Phi(f) = \Phi(g)$ whenever $g(t) = t \cdot f(t)$ and both f, g are in $AC^{(\nu+1)}$. Since $AC^{(\nu+1)}$ is dense in $AC^{(\nu)}$, Φ becomes an $AC^{(\nu)}$ calculus for A. Moreover, $\Phi(AC^{(\nu)})X$ is dense in X because, by (4.4), for any $x \in X$

$$\Phi(e_t)x = t^{\nu+1} \int_0^\infty e^{-tu} G^\nu(u)x\,du = a^t x$$

tends to x as $t \to 0^+$. ∎

C. *The Gauss and Poisson Semigroups on R^n*

As is known, if Δ is the Laplace operator

$$\Delta = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \cdots - \frac{\partial^2}{\partial x_n^2}$$

then $-\Delta$ generates a convolution semigroup $\{a^z(\cdot)\}_{\Re z>0}$ in $L^1(\mathbf{R}^n)$, called *the Gauss semigroup*, with

$$\tilde{a}^z(x) = (4\pi z)^{-n/2}\, e^{-|x|^2/(4z)}.$$

For any p, $1 \leqslant p < \infty$, this semigroup extends to a (C_0) holomorphic semigroup $\{a^z\}_{\Re z>0}$ of operators on $L^p(\mathbf{R}^n)$ (here and later we write a and $\tilde{a}$ to distinguish operators from their convolution kernels). Since $\|\tilde{a}^z\|_1 = (|z|/\Re z)^{n/2}$ and $\|(\tilde{a}^z)^\wedge\|_\infty = 1$, we have $\|a^z\| \leqslant (|z|/\Re z)^{n|1/p-1/2|}$ by the Interpolation Theorem. This implies that the semigroup is subhomogeneous of order $\alpha = n\,|1/p - 1/2|$.

Each of the operators $G^\nu(u) = R_u^\nu(\Delta)$ is a convolution with a function (say g_u), whose Fourier transform is

$$\widehat{g_u}(y) = \frac{1}{2\pi i} \int_{\Re z=1} \frac{\widehat{\tilde{a}^z}(y)}{z^{\nu+1}}\, e^{uz}\,dz = \frac{1}{2\pi i} \int_{\Re z=1} \frac{e^{-4\pi^2 z\,|y|^2}}{z^{\nu+1}}\, e^{uz}\,dz$$

$$= \frac{1}{\Gamma(\nu+1)}\, (u - 4\pi^2\,|y|^2)_+^\nu.$$

Therefore (see [SW] IV, Theorem 4.15) g_u is the Bochner–Riesz mean

$$g_u(x) = 2^{-n/2+\nu} \pi^{-n/2} (\sqrt{u}/|x|)^{n/2+\nu}\, J_{n/2+\nu}(\sqrt{u}\,|x|).$$

The known estimate $O(|x|^{-1/2})$ for Bessel functions ([SW] IV, Lemma 3.11) implies that all operators $G^\nu(u)$ are bounded on $L^1(\mathbf{R}^n)$ (g_u are integrable) whenever $\nu > (n-1)/2$. The integral (4.2) is however absolutely convergent only when $\nu > n/2$. It means that direct application of Theorem 6.2 or Theorem 6.3 would give weaker functional calculi than expected (for example from Proposition 7.5). As we will see, this can be easily omitted by passing to the Poisson semigroup. The L^p-boundedness of the Bochner–Riesz kernel $R_u^\nu(\Delta)$ in case $1 < p < \infty$ is a delicate matter (the *Bochner–Riesz conjecture*). It is discussed in [F].

Now let $\sqrt{\Delta}$ be the square root of Δ in the sense that $\sqrt{\Delta} = \Delta^{1/2}$ (Sect. 7.A, [Y, p. 268], [G, p. 63]). The operator $-\sqrt{\Delta}$ generates a convolution semigroup $\{\tilde{p}^z(\cdot)\}\,\Re z > 0$ in $L^1(\mathbf{R}^n)$, called *the Poisson semigroup*, where

$$\tilde{p}^z(x) = \Gamma\left(\frac{n+1}{2}\right)\pi^{-(n+1)/2}\,\frac{z}{(z^2+|x|^2)^{(n+1)/2}}.$$

We have $\|\tilde{p}^z\|1 \leqslant C\log(2+|z|/\Re z)$ when $n=1$ and $\|\tilde{p}^z\|_1 \leqslant C(|z|/\Re z)^{(n-1)/2}$ when $n>1$. So it has a lower rate of growth than the Gauss semigroup. The norm estimate is a special case of the general inequality

$$\int_0^\infty \frac{|z|r^{\alpha-1}}{|z^2+r^2|^\beta}\,dr \leqslant C_{\alpha,\beta}(\Re z)^{1-\beta}\,|z|^{\alpha-\beta}, \qquad \Re z > 0 \tag{7.2}$$

valid for any $\alpha > 0$ and $\beta > \max\{1, \alpha/2\}$.

To see the uniform boundedness in z, $\Re z > 0$, of

$$|z|^{\beta-\alpha+1}(\Re z)^{\beta-1}\int_0^\infty \frac{r^{\alpha-1}}{|z^2+r^2|^\beta}\,dr$$

note that $|z^2+r^2|^2 = (|z|^2-r^2)^2 + 4r^2(\Re z)^2$. After change of the variable $r = |z|t$ it becomes

$$\int_0^\infty \lambda\, t^{\alpha-1}[\lambda^2(1-t^2)^2+4t^2]^{-\beta/2}\,dt, \qquad \lambda = |z|/\Re z.$$

Split $\int_0^\infty$ into $\int_{|t-1|\leqslant 1/2} + \int_{|t-1|>1/2}$. Then the first integral is dominated by

$$2^\alpha\int_{1/2}^{3/2}\lambda t[\lambda^2(1-t^2)^2+1]^{-\beta/2}\,dt \leqslant 2^\alpha\int_{-\infty}^{\infty}(u^2+1)^{-\beta/2}\,du,$$

and the second one by $\int_{|t-1|>1/2} t^{\alpha-1}\,|1-t^2|^{-\beta}\,dt$, since $\lambda^{1-\beta} \leqslant 1$.

As an application of Theorem 6.3 we get the following Bernstein's type result (cf. [P, Théorème 2.1]).

PROPOSITION 7.6. *Let f be a radial function on R^n, i.e. $f(x) = f_0(|x|)$ for $x \in R^n$. If $\nu > n/2$ and $f_0 \in AC^{(\nu)}_{2,1}$ then the Fourier transform $\hat{f}$ of f lies in $L^1(\mathbf{R}^n)$ and*

$$\|\hat{f}\|_1 \leqslant C\, \|f_0\|_{(\nu),\,2,\,1}.$$

Proof. Assume $f_0 \in \mathscr{E}$. Then $f_0((1/2\pi)\sqrt{\Delta})$ is a bounded operator on $L^1(\mathbf{R}^n)$. Since the integrals (4.1) and (4.2) are absolutely convergent $f_0((1/2\pi)\sqrt{\Delta})$ is in fact a convolution with an $L^1(\mathbf{R}^n)$ function $\tilde{f}$ and $\|f_0((1/2\pi)\sqrt{\Delta})\| = \|\tilde{f}\|_1$. But $(\tilde{f})^\wedge = f$, so the Proposition follows from Theorem 6.3. ∎

The following result is to be compared with [D1, pp. 174,175], [G, p. 62].

PROPOSITION 7.7. *Let Δ be the Laplace operator on $\mathbf{R}^n$. For any $\theta > 0$ the operator $-\Delta^\theta$ generates in $L^1(\mathbf{R}^n)$ a holomorphic convolution (C_0) semigroup which is homogeneous with growth exponent α_θ of the semigroup satisfying the following inequality $(n-1)/2 \leqslant \alpha_\theta \leqslant n/2$.*

Proof. As we have seen in Section 7.A, the semigroup generated by the operator $-\Delta^\theta = -(\sqrt{\Delta})^{2\theta}$ on $L^1(\mathbf{R}^n)$ has the form $\{e_z(\sqrt{\Delta})\}_{\Re z > 0}$, where $e_z(u) = \exp(-z\, u^{2\theta})$. Clearly each of operators $e_z(\sqrt{\Delta})$ is a convolution with an $L^1(\mathbf{R}^n)$ function $\tilde{e}_z(\sqrt{\Delta})$ and $\|\tilde{e}_z(\sqrt{\Delta})\|_1 = \|e_z(\sqrt{\Delta})\|$. Since the growth exponent of the Poisson semigroup is $(n-1)/2$, for any $\nu > (n-1)/2$ by Theorem 6.3 and Proposition 3.9 we obtain

$$\begin{aligned}
\|e_z(\sqrt{\Delta})\| &\leqslant C\, \|e_z\|_{(\nu+1/2),\,2,\,1} \\
&\leqslant C' \int_0^\infty \left[\int_y^{2y} |x^\nu z^{\nu+1/2} e^{-zu}|^2\, dx \right]^{1/2} y^{-1}\, dy \\
&\leqslant C''(|z|/\Re z)^{\nu+1/2}.
\end{aligned}$$

This implies that $\alpha_\theta \leqslant n/2$. If for some θ_0 we had $\alpha_{\theta_0} < (n-1)/2$, then arguing as above with $-\Delta^{\theta_0}$ instead of $\sqrt{\Delta}$ we would obtain $\alpha_\theta < n/2$ for any $\theta > 0$, which is impossible as $\alpha_1 = n/2$. Consequently $(n-1)/2 = \alpha_{1/2} \leqslant \alpha_\theta \leqslant \alpha_1 = n/2$. ∎

Remark. Generally in Theorem 6.3 it is not possible to use lower orders of derivation than $\alpha + 1/2$ for the operating functions, for otherwise, doing as in the proof of Proposition 7.7 we would get a growth exponent for the Gauss semigroup in $L^1(\mathbf{R}^n)$ strictly less than $n/2$.

Similar results can be obtained for the convolution Banach algebra $L^1(\mathbf{R}^n, \delta)$ of all functions integrable with respect to the weight $(1+|x|^2)^\delta$,

$\delta > 0$. Although the semigroup generated by Δ is not subhomogeneous it satisfies (G_α) with $\alpha = n/2 + 2\delta$, and since $\|G^\nu(u)\| \leqslant Cu^\nu(1+u^{-\delta})$ for $\nu > \alpha$, the condition

$$\int_0^\infty |W^{\nu+1} f(u)|\, u^\nu (1+u^{-\delta})\, du < \infty$$

suffices for a function f to operate on Δ into $L1^{(}\mathbf{R}^n, \delta)$.

We claim that if θ is a natural number or $\theta > \delta$ then $-\Delta^\theta$ generates in $L^1(\mathbf{R}^n, \delta)$ a holomorphic (C_0) convolution semigroup which satisfies (G_{m+1}) for any integer $m > \alpha$. Indeed, the chain rule for multiple derivation applied to the function $f(u) = \exp(-zu^\theta)$ gives

$$\frac{d^{m+1}}{du^{m+1}} \exp(-zu^\theta)$$

$$= (m+1)! \sum_{k=1}^{m+1} (-z)^k \exp(-zu^\theta) \sum \prod_{r=1}^{m+1} \frac{1}{p_r\,!} \left[\binom{\theta}{r} u^{\theta - r} \right]^{p_r},$$

where $\sum$ extends over all combinations of nonnegative integer values of $p_1, p_2, ..., p_{m+1}$, such that

$$\sum_{r=1}^{m+1} p_r = k \qquad \text{and} \qquad \sum_{r=1}^{m+1} r p_r = m+1.$$

It follows that $u^m(1+u^{-\delta})\, W^{m+1} f(u)$ is rapidly decreasing at infinity and at zero is $O(u^{m-\delta})$ if θ is a natural number and $O(u^{\theta-\delta-1})$ otherwise.

Finally let us signify a possibility of application of the results of Section 6 to Laplacian with potential on $\mathbf{R}^n$. Let V be a real-valued, measurable function on $\mathbf{R}^n$, with V_+ a *Kato perturbation* and $V_- \in L^\infty(\mathbf{R}^n)$. Then there exists $\omega \in \mathbf{R}$ such that the operator $A = \Delta + V - \omega$ generates a holomorphic (C_0) semigroup which satisfies (HG_n) ([deL3, p. 76]). Thus bounded operators $f(A)$ on $L^p(\mathbf{R}^n)$, $1 \leqslant p < \infty$, or $C_0(\mathbf{R}^n)$ can be accordingly constructed for $f \in AC^{(\nu+1)}$ or $f \in AC^{(\nu+1/2)}_{2,1}$, $\nu > n$, whereas it would be, seemingly, a difficult task to get such $f(A)$ directly.

D. *Gauss and Poisson Semigroups on Stratified Lie Groups*

Let G be a stratified nilpotent Lie group of homogeneous dimension n and let $L = -(X_1^2 + X_2^2 + ... + X_k^2)$ be the homogeneous sublaplacian on G (we refer to [FS] for basic notation and results). Let $\int_0^\infty \lambda\, dE(\lambda)$ be the spectral resolution of $\sqrt{L}$ on $L^2(G)$. Then for any bounded and continuous function f on $[0, \infty)$ the operator $f(\sqrt{L}) = \int_0^\infty f(\lambda)\, dE(\lambda)$ is a convolution on the right with a distribution $\tilde{f}(\sqrt{L})(\cdot)$. Let us recall here a result due to Sikora [S, Lemma 2] based on the finite propagation speed of the operator $\cos(t\sqrt{L})$. It says that

$$\int_G |x|^\theta \, |\tilde{f}(\sqrt{L})(x)|^2 \, dx \leqslant \int_{R^n} |x|^\theta \, |\tilde{f}(\sqrt{\Delta})(x)|^2 \, dx \tag{7.3}$$

for any $\theta > 0$. Here $|\cdot|$ denotes in the first integral the distance to e in the Caratheodory metric of G defined by $X_1, X_2, ..., X_k$ and the Euclidean distance in the second integral. Also Δ is the usual Laplacian on $\mathbf{R}^n$.

PROPOSITION 7.8. *Let L denote the sub-Laplacian on a stratified Lie group G. Then the homogeneous convolution (C_0) semigroup (the Poisson semigroup) $\{\tilde{p}^z(\sqrt{L})(\cdot)\}_{\Re z > 0}$ generated by $-\sqrt{L}$ lies in $L^1(G)$ and satisfies*

$$\|\tilde{p}^z(\sqrt{L})(\cdot)\|_1 \leqslant C(|z|/Re)^{n/2}, \tag{7.4}$$

where n is the homogeneous dimension of G.

Proof. Choose θ so that $n < \theta < n+2$. Then by the Sobolev inequality, by (7.3) and by (7.2) we obtain

$$\begin{aligned} \|\tilde{p}^z(\sqrt{L})\|_1^2 &\leqslant C_1 \int_G |\tilde{p}^z(\sqrt{L})(x)|^2 \, (1 + |x|^\theta) \, dx \\ &\leqslant C_1 \int_{R^n} |\tilde{p}^z(\sqrt{\Delta})(x)|^2 \, (1 + |x|^\theta) \, dx \\ &= C_2 \int_0^\infty \frac{|z|^2 \, r^{n-1}(1 + r^\theta)}{|z^2 + r^2|^{n+1}} \, dr \leqslant (\Re z)^{-n} \, (C_3 + C_4 \, |z|^\theta). \end{aligned}$$

But $\{\tilde{p}^z(\sqrt{L})\}$ is a homogeneous semigroup. Thus $\|\tilde{p}^{tz}(\sqrt{L})\|_1 = \|\tilde{p}^z(\sqrt{L})\|_1$ for any $t > 0$. In particular for $t = |z|^{-1}$ we obtain (7.4). ∎

Proposition 7.8, Theorem 6.3, and Proposition 3.9 applied to $\sqrt{L}$ give the following.

COROLLARY 7.9. *Let G be a stratified Lie group of homogeneous dimension n, and let $g(L) = \int_0^\infty g(\lambda) \, dE(\lambda)$ be, for $g \in L^\infty([0, \infty))$, the functional calculus associated to the spectral decomposition of the sub-Laplacian L on G. Then $f(L) \in L^1(G)$ with $\|f(L)\|_1 \leqslant C \, \|f\|_{(\nu), 2, 1}$ for every $f \in AC^{(\nu)}_{2, 1}$ and $\nu > (n+1)/2$.* ∎

Of course results about fractional powers L^θ can be given in the same way as done in Sect. 7C.

The above corollary by interpolation extends to a multiplier theorem for $L^p(G)$, $1<p<\infty$. It is however not new. The earliest result of that type was obtained by Hulanicki and Stein (cf. [FS, pp. 208–215]). The strongest one is by Christ [C] and by Mauceri and Meda [MM]. In particular, it is shown there that if $\nu>n/2$ and $f\in AC^{(\nu)}_{2,\infty}$ then the operator $f(L)$ is of weak type (1,1) and bounded on each $L^p(G)$, $1<p<\infty$.

Remark. For the Gauss semigroup (generated by L), by the same method as in Proposition 7.8, for any $\theta>n$ we obtain an estimate

$$\|\tilde{h}^z\|_1 \leqslant C_\theta(|z|/Rez)^{(n+\theta)/4},$$

which implies that the growth exponent α of the semigroup is at most $n/2$. The same follows from [C], and has been proven in [Du] by other method. The estimate however does not seem to be sharp. For generalized Heisenberg groups of homogeneous dimension $2d+2k$, where k is the dimension of the center, J. Randall [R] obtained a better estimate $\|\tilde{h}^z\|_1 \leqslant C(|z|/\Re z)^\alpha$ with $\alpha=d+(k+3)/2$ for k odd, and $d+(k+4)/2$ for k even. Let us also remark a result of Hebisch [H] which implies that for generalized Heisenberg groups the growth exponent is $d+k/2$, half of the Euclidean dimension of the group.

For estimates on vertical lines of the right-hand half-plane of $\mathbf{C}$, of holomorphic semigroups on more general L^p spaces, see [D2] and references therein.

E. *Bochner–Riesz Operators and Approximation*

Suppose that A is a closed, densely defined operator on a Banach space X, not related a priori to a holomorphic semigroup. Let $\beta\geqslant 0$ and assume that the Bochner–Riesz mean R^ν_u, for $\nu>\beta$ and $u\geqslant 0$, acts somehow on A so that it defines a suitable Bochner–Riesz operator $R^\nu_u(A)$ on X in the sense that, at least $u\to R^\nu_u(A)$ is a continuous function from $[0,\infty)$ to $B(X)$ and

$$\Gamma(\nu+1)\lim_{u\to\infty} u^{-\nu}R^\nu_u(A)x=x \tag{7.5}$$

for every $x\in X$ (*summability property*).

The usual way to present Bochner–Riesz operators consists of assuming that there exists a Hilbert space H for which $H\cap X$ is dense in X, A has a spectral decomposition on H, so that $R^\nu_u(A)\in B(H)$, and $R^\nu_u(A)$ extends to $B(X)$ via $H\cap X$. Under this form, it has been observed that operators $R^\nu_u(A)$ and the $AC^{(\nu+1)}$ calculus $f\to\int_0^\infty W^{\nu+1}f(u)\,R^\nu_u(A)\,du$ that they define have applications in approximation theory ([BNT], [P, p. 191], and references therein).

It is clear that under a $AC^{(\nu+1)}$ calculus for A the semigroup $\{a^z\}_{\Re z>0}$, where $a^z = z^{\nu+1}\int_0^\infty e^{-zu} R_u^\nu(A)\,du$, satisfies (C_0) and (G_α) with $\alpha=\beta+1$. Conversely, if $-A$ generates a holomorphic (C_0) semigroup $\{a^z\}_{\Re z>0}$ in $B(X)$ which satisfies (G_α) then (7.5) holds for $\nu>\alpha$ with $R_u^\nu(A)$ defined by (4.2), for then

$$\Gamma(\nu+1)\,u^{-\nu}R_u^\nu(A)\,a^1x = \frac{\Gamma(\nu+1)}{2\pi i}\int_{\Re z=1}\frac{a^{1+z/u}x}{z^{\nu+1}}\,e^z\,dz$$

$$\xrightarrow[u\to\infty]{} \frac{\Gamma(\nu+1)}{2\pi i}\int_{\Re z=1}\frac{e^z}{z^{\nu+1}}\,dz\cdot a^1x = a^1x$$

for every $x\in X$, because of the dominated convergence theorem and (4.3). Now, since the family of operators $\{u^{-\nu}R_u^\nu(A)\}_{u\geqslant 0}$ is uniformly bounded in u, it is enough to recall that the set $\{a^1x\colon x\in X\}$ is dense in X.

Therefore Bochner–Riesz operators and holomorphic semigroups are equivalent objects, essentially. This fact might be applied to approximation theory along the same lines as in [BNT], for instance. The approach based on the semigroup $\{a^z\}_{\Re z>0}$ rather than the Bochner–Riesz means has the advantage that no appeal to underlying Hilbert spaces or spectral decompositions of the infinitesimal generator is required, and, above all, that Theorem 6.3 allows us to improve the degree of smoothness of the functional calculus, whenever the growth exponent α is taken to be the best possible. On the contrary, it is usually more complicated to find the "best" α than the "best" β, in practice.

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